Problem 6.3

- Use $G(\alpha,\beta;E) = \sum_m \frac{\langle\alpha|m\rangle\langle m|\beta\rangle}{E-\varepsilon_m+i\eta} + \int d\mu\, \frac{\langle\alpha|\mu\rangle\langle\mu|\beta\rangle}{E-\varepsilon_\mu+i\eta}$ $\quad G^{(0)}(\alpha,\beta;E) = \frac{\delta_{\alpha,\beta}}{E-\varepsilon_\alpha+i\eta}$
- $\lim_{E\to\varepsilon_n}(E-\varepsilon_n)\{G = G^{(0)} + G^{(0)}\ V\ G\}$ or $\lim_{E\to\varepsilon_n}(E-\varepsilon_n)\{G = G^{(0)} + G\ V\ G^{(0)}\}$

$$\langle\alpha|n\rangle = \sum_{\gamma\delta}\langle\alpha|\,\frac{1}{\varepsilon_n - H_0}\,|\gamma\rangle\,\langle\gamma|\,V(\varepsilon_n)\,|\delta\rangle\,\langle\delta|n\rangle \qquad \langle n|\beta\rangle = \sum_{\gamma\delta}\langle n|\gamma\rangle\,\langle\gamma|\,V(\varepsilon_n)\,|\delta\rangle\,\langle\delta|\,\frac{1}{\varepsilon_n - H_0}\,|\beta\rangle$$

- Normalization: near pole of propagator

$$G(\alpha,\beta;E\to\varepsilon_n) \Rightarrow \frac{\langle\alpha|n\rangle\langle n|\beta\rangle}{E-\varepsilon_n} + f_{\alpha\beta}(\varepsilon_n)$$

- Smooth behavior at this energy of other terms -->

$$\frac{G^{(0)}(E)V(E)}{E-\varepsilon_n} = \frac{G^{(0)}(\varepsilon_n)V(\varepsilon_n)}{E-\varepsilon_n} + \frac{\partial G^{(0)}V}{\partial E}\Big|_{\varepsilon_n}$$

- Insert in

$$G(\alpha,\beta;E) = G^{(0)}(\alpha,\beta;E) + \sum_{\gamma,\delta} G^{(0)}(\alpha,\gamma;E)\,\langle\gamma|\,V\,|\delta\rangle\,G(\delta,\beta;E)$$

Result

- yielding

$$\frac{\langle \alpha|n\rangle\langle n|\beta\rangle}{E-\varepsilon_n} \quad + \quad f_{\alpha\beta}(\varepsilon_n) = G^{(0)}(\alpha,\beta;\varepsilon_n)$$

$$+ \quad \sum_{\gamma\delta} G^{(0)}(\alpha,\gamma;E)\,\langle\gamma|\,V(E)\,|\delta\rangle \left\{ \frac{\langle \delta|n\rangle\langle n|\beta\rangle}{E-\varepsilon_n} + f_{\delta\beta}(\varepsilon_n)\right\}$$

$$= \quad G^{(0)}(\alpha,\beta;\varepsilon_n) + \sum_{\gamma\delta} G^{(0)}(\alpha,\gamma;\varepsilon_n)\,\langle\gamma|\,V(\varepsilon_n)\,|\delta\rangle\,\frac{\langle \delta|n\rangle\langle n|\beta\rangle}{E-\varepsilon_n}$$

$$+ \quad \sum_{\gamma\delta} G^{(0)}(\alpha,\gamma;\varepsilon_n)\,\langle\gamma|\,V(\varepsilon_n)\,|\delta\rangle\, f_{\delta\beta}(\varepsilon_n)$$

$$+ \quad \sum_{\gamma\delta} \left.\frac{\partial G^{(0)}(\alpha,\gamma;E)\,\langle\gamma|\,V(E)\,|\delta\rangle}{\partial E}\right|_{\varepsilon_n} \langle \delta|n\rangle\langle n|\beta\rangle$$

continued

- Resulting in

$$f_{\alpha\beta}(\varepsilon_n) = G^{(0)}(\alpha,\beta;\varepsilon_n) + \sum_{\gamma\delta} G^{(0)}(\alpha,\gamma;\varepsilon_n)\langle\gamma|V(\varepsilon_n)|\delta\rangle f_{\delta\beta}(\varepsilon_n) + \sum_{\gamma\delta} \left.\frac{\partial G^{(0)}(\alpha,\gamma;E)\langle\gamma|V(E)|\delta\rangle}{\partial E}\right|_{\varepsilon_n} \langle\delta|n\rangle\langle n|\beta\rangle$$

- Next step

$$\sum_\alpha \langle n|(\varepsilon_n - H_0)|\alpha\rangle f_{\alpha\beta}(\varepsilon_n) = \sum_\alpha \langle n|(\varepsilon_n - H_0)|\alpha\rangle G^{(0)}(\alpha,\beta;\varepsilon_n) + \sum_{\gamma\delta}\sum_\alpha \langle n|(\varepsilon_n - H_0)|\alpha\rangle G^{(0)}(\alpha,\gamma;\varepsilon_n)\langle\gamma|V(\varepsilon_n)|\delta\rangle f_{\delta\beta}(\varepsilon_n) + \sum_{\gamma\delta}\sum_\alpha \langle n|(\varepsilon_n - H_0)|\alpha\rangle \left.\frac{\partial G^{(0)}(\alpha,\gamma;E)\langle\gamma|V(E)|\delta\rangle}{\partial E}\right|_{\varepsilon_n} \langle\delta|n\rangle\langle n|\beta\rangle$$

- Consider

$$\langle n|\beta\rangle = \sum_\alpha \langle n|(\varepsilon_n - H_0)|\alpha\rangle G^{(0)}(\alpha,\beta;\varepsilon_n)$$

- and

$$\begin{aligned}
\sum_\alpha \langle n|(\varepsilon_n - H_0)|\alpha\rangle f_{\alpha\beta}(\varepsilon_n) &= \sum_\alpha \langle n|\alpha\rangle(\varepsilon_n - \varepsilon_\alpha) f_{\alpha\beta}(\varepsilon_n) \\
&= \sum_{\alpha\gamma} \langle n|\gamma\rangle\langle\gamma|V(\varepsilon_n)|\alpha\rangle f_{\alpha\beta}(\varepsilon_n) \\
&? \sum_{\gamma\delta}\sum_\alpha \langle n|(\varepsilon_n - H_0)|\alpha\rangle G^{(0)}(\alpha,\gamma;\varepsilon_n)\langle\gamma|V(\varepsilon_n)|\delta\rangle f_{\delta\beta}(\varepsilon_n) \\
&= \sum_{\gamma\delta} \langle n|\gamma\rangle\langle\gamma|V(\varepsilon_n)|\delta\rangle f_{\delta\beta}(\varepsilon_n)
\end{aligned}$$

continued

- So now we obtain

$$
\begin{aligned}
0 &= \langle n|\beta\rangle \\
&+ \sum_{\gamma\delta}\sum_{\alpha} \langle n|\left(\varepsilon_n - H_0\right)|\alpha\rangle \left.\frac{\partial G^{(0)}(\alpha,\gamma;E)\,\langle\gamma|\,V(E)\,|\delta\rangle}{\partial E}\right|_{\varepsilon_n} \langle\delta|n\rangle\langle n|\beta\rangle
\end{aligned}
$$

- or

$$
\begin{aligned}
0 &= 1 \\
&+ \sum_{\gamma\delta}\sum_{\alpha} \langle n|\left(\varepsilon_n - H_0\right)|\alpha\rangle \left.\frac{\partial G^{(0)}(\alpha,\gamma;E)\,\langle\gamma|\,V(E)\,|\delta\rangle}{\partial E}\right|_{\varepsilon_n} \langle\delta|n\rangle
\end{aligned}
$$

- So

$$
\begin{aligned}
0 &= 1 \\
&- \sum_{\gamma\delta}\sum_{\alpha}(\varepsilon_n - \varepsilon_\alpha)\langle n|\alpha\rangle\delta_{\alpha\gamma}\frac{1}{(\varepsilon_n - \varepsilon_\alpha)^2}\,\langle\gamma|\,V(\varepsilon_n)\,|\delta\rangle\,\langle\delta|n\rangle \\
&+ \sum_{\gamma\delta}\langle n|\gamma\rangle\,\langle\gamma|\left.\frac{\partial V(E)}{\partial E}\right|_{\varepsilon_n}|\delta\rangle\,\langle\delta|n\rangle \\
&= 1 - \sum_{\gamma\delta}\langle n|\gamma\rangle\frac{1}{\varepsilon_n - \varepsilon_\gamma}\,\langle\gamma|\,V(\varepsilon_n)\,|\delta\rangle\,\langle\delta|n\rangle \\
&+ \sum_{\gamma\delta}\langle n|\gamma\rangle\,\langle\gamma|\left.\frac{\partial V(E)}{\partial E}\right|_{\varepsilon_n}|\delta\rangle\,\langle\delta|n\rangle
\end{aligned}
$$

Finally

- Yielding the final result

$$
\begin{aligned}
1 &= \sum_{\gamma} \langle n|\gamma\rangle \langle\gamma|n\rangle \\
&- \sum_{\gamma\delta} \langle n|\gamma\rangle \left\langle\gamma\right| \left.\frac{\partial V(E)}{\partial E}\right|_{\varepsilon_n} \left|\delta\right\rangle \langle\delta|n\rangle
\end{aligned}
$$