Problem set 9

1. The allowed values of the energy is $E = \frac{k_I^2 \hbar^2}{2 m}$.

2. $k_{II} = \sqrt{\frac{2 m V_0}{\hbar^2} - k_I^2} = k_I \tan\left(\frac{k_I a}{2}\right)$.

Hence $\frac{k_I a}{2} \tan\left(\frac{k_I a}{2}\right) = \frac{a}{2} \sqrt{\frac{2 m V_0}{\hbar^2} - k_I^2}$.

Make the substitution $\Upsilon = \frac{a \sqrt{2 m V_0}}{2 \hbar}$, $\theta = \frac{k_1 a}{2}$,

and we arrive at the desired expression $\theta \tan\theta = \sqrt{\Upsilon^2 - \theta^2}$.

In[1]:=
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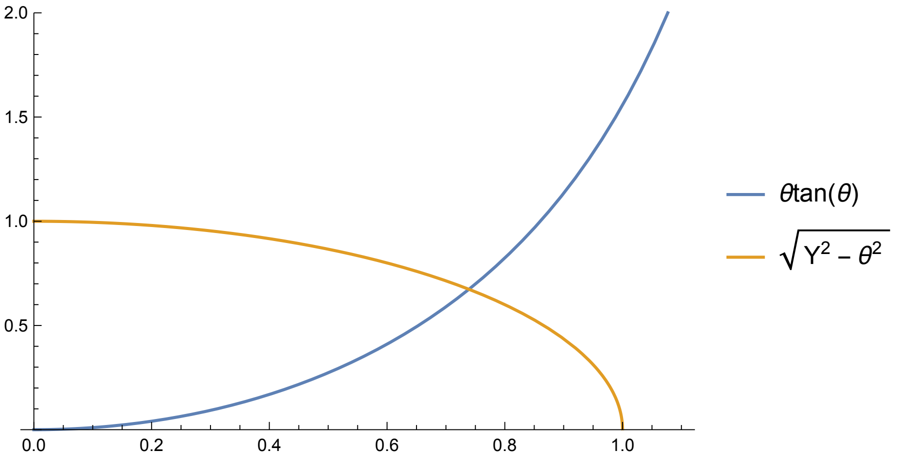
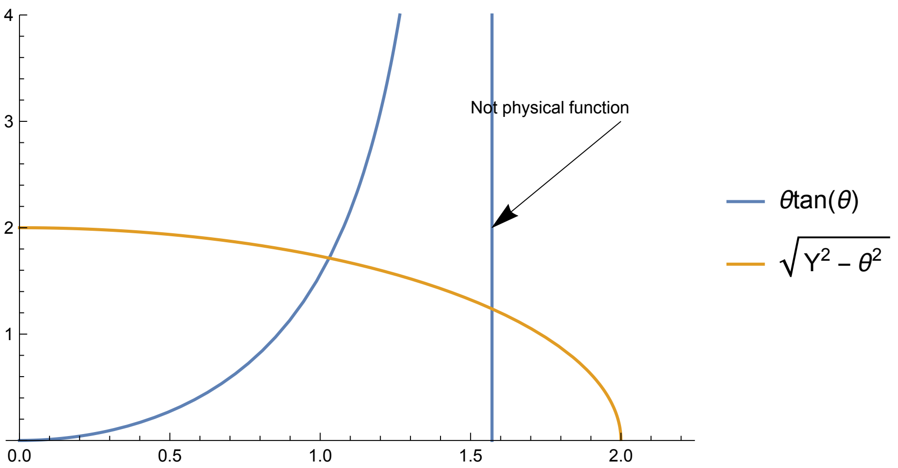
plot$even[Υ_] = Plot[{θ Tan[θ], Sqrt[Υ^2 - θ^2]}, {θ, 0, 1.1 Υ}, PlotRange → {0, 2 Υ},
    PlotLegends → {"θtan(θ)", "Sqrt[Υ^2 - θ^2]"}, Epilog → {Arrow[{{2, 3}, {Pi / 2, 2}}],
      Text["Not physical function", {1.5, 3}, {-1, -1}]}];
plot$even[1]
plot$even[2]
plot$even[6]
```

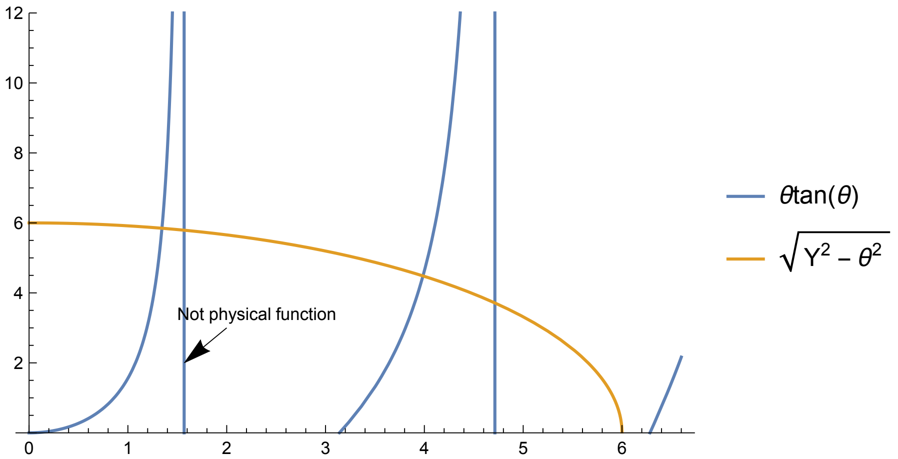
Out[2]=



Out[3]=



Out[4]=



For $\Upsilon = 1$, there is only one even bound state;
For $\Upsilon = 2$, there is only one even bound state;
For $\Upsilon = 6$, there are two even bound states.

3. The wavefunction and its derivative are continuous at $\frac{a}{2}$. Hence we have

$$A\sin\left(k_I \frac{a}{2}\right) = Ce^{-k_{II}\frac{a}{2}} \qquad (1)$$

$$Ak_I \cos\left(k_I \frac{a}{2}\right) = -k_{II}\, Ce^{-k_{II}\frac{a}{2}} \qquad (2)$$

Divide equation (1) by equation (2),

$\frac{1}{k_I}\tan\left(k_I \frac{a}{2}\right) = -\frac{1}{k_{II}}$, where $k_I^2 + k_{II}^2 = \frac{2\,mV_0}{\hbar^2}$.

Boundary condition at $x = -\frac{a}{2}$: $-A\sin\left(k_I \frac{a}{2}\right) = C_I\, e^{k_I\frac{a}{2}}$;

boundary condition at $x = \frac{a}{2}$: $A\sin\left(k_I \frac{a}{2}\right) = C_{III}\, e^{-k_{II}\frac{a}{2}}$. Therefore,

$C_I = -C_{III}$, the value of C for the solution in region I is the negative of the value for the solution in region III.

$$k_I \cot\left(k_I \frac{a}{2}\right) = -k_{II} = -\sqrt{\frac{2\,mV_0}{\hbar^2} - k_I^2}$$

$$\frac{ak_I}{2}\cot\left(k_I \frac{a}{2}\right) = -\frac{a}{2}\sqrt{\frac{2\,mV_0}{\hbar^2} - k_I^2}$$

If we make the substitution $\Upsilon = \frac{a\sqrt{2\,mV_0}}{2\,\hbar}$,

$\theta = \frac{ak_I}{2}$, then we arrive at $-\theta\cot\theta = \sqrt{\Upsilon^2 - \theta^2}$.

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In[8]:= plot$odd[Υ_] = Plot[{-θ Cot[θ], √(Υ^2 - θ^2)}, {θ, 0, 1.1 Υ},
    PlotRange → {0, 2 Υ}, PlotLegends → {"-θcot(θ)", "√(Υ^2 - θ^2)"}];
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In[9]:=
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plot$odd[1]
plot$odd[2]
plot$odd[6]
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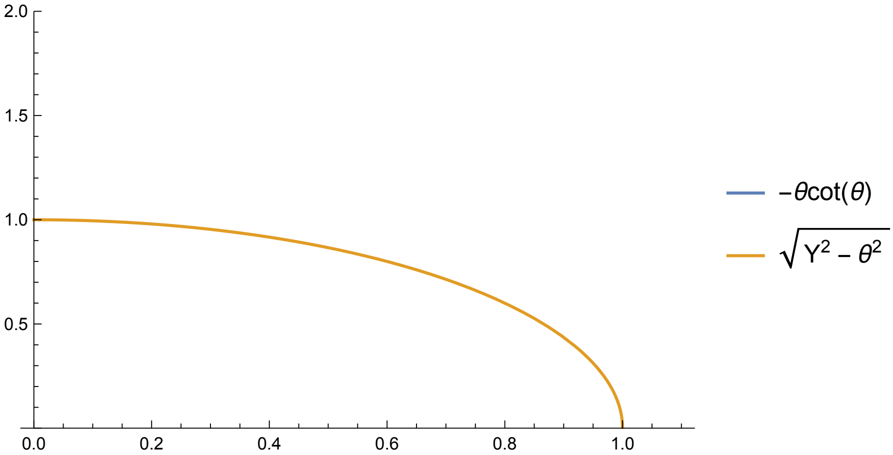
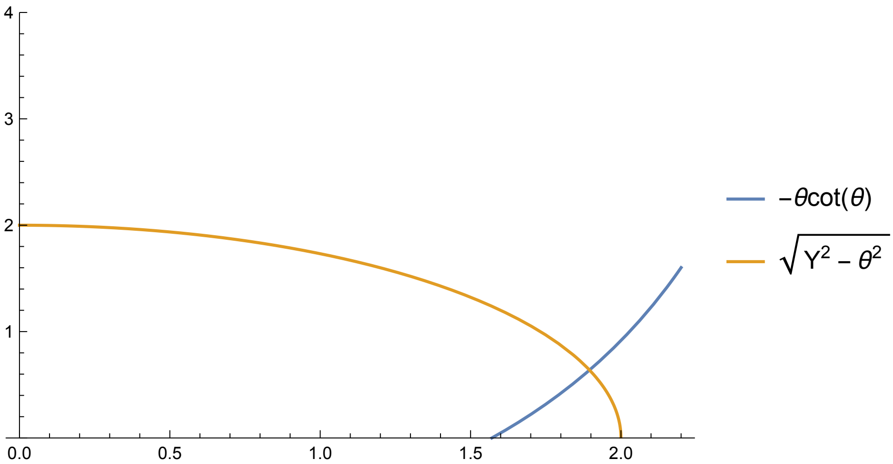
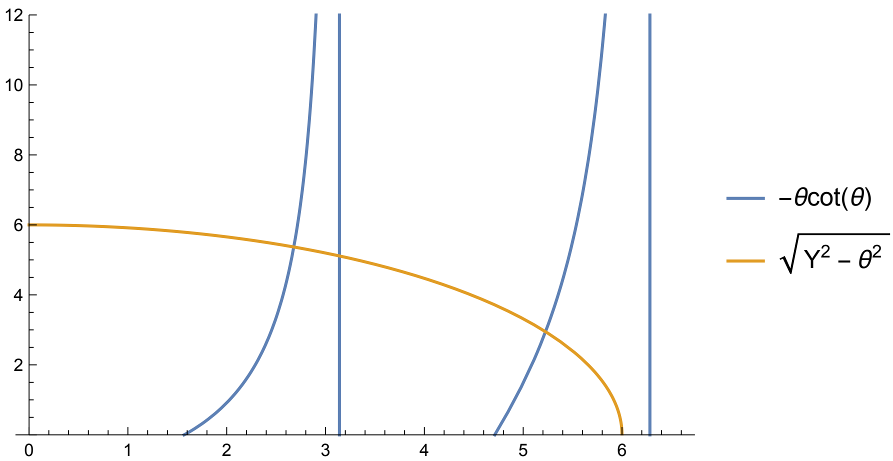
Out[9]=



Out[10]=



Out[11]=



For ϒ = 1, there is no odd bound state;
For ϒ = 2, there is only one odd bound state;
For ϒ = 6, there are two odd bound states.

4. If $\Upsilon = \frac{a\sqrt{2\,mV_0}}{2\,\hbar} \to 0$, then the depth of the well $V_0 \to 0$.
However as long as the depth of the well is not exactly 0,
there should always be (at least) one solution. This means that its
binding energy goes to zero and its spatial extent goes to infinity.

5. For large Υ, $\theta\tan\theta = \sqrt{\Upsilon^2 - \theta^2} \sim \Upsilon$.
The energies of these states are E =

$$\frac{k_I^2\,\hbar^2}{2\,m} = \frac{\left(\frac{2\,\theta}{a}\right)^2 \hbar^2}{2\,m} = \frac{2\left(j + \frac{1}{2}\right)^2 \pi^2\,\hbar^2}{ma^2} = \frac{(2\,j+1)^2\,\pi^2\,\hbar^2}{2\,ma^2},$$

where j is an integer, i.e. $(2\,j+1)$ is an odd number.
Recall that the eigenenergies corresponding to
the even eigenstates (that is, the cos function solutions)
of the infinite well are $\frac{\hbar^2}{2\,m}\left(\frac{n\pi}{a}\right)^2$, where n is an odd number.
Therefore, when $\Upsilon \to \infty$ we restore the infinite well case.
This makes sense since if $\Upsilon = \frac{a\sqrt{2\,mV_0}}{2\,\hbar} \to \infty$ at fixed a, then we must have $V_0 \to \infty$.