Problem set 8

(* 1. The point of this problem is to illustrate that if a particle's position is well known, its momentum is not. Initially we know the particle is located inside the nucleus, meaning that $\Delta r=7.8\times10^{-15}$m. Using the uncertainty relation we get $\Delta p \geqslant \frac{\hbar}{2\Delta r}$. The uncertainty in the velocity is $\Delta v=\frac{\Delta p}{m}=7.4\times10^{9}$, which is larger than the speed of light. This illustrates the fact that we do not know anything about the momentum. The possible energy of the electron is $\Delta E=\frac{(\Delta p)^2}{2m}=2.49\times10^{-11}$J=156MeV.

Here you notice that this energy value is much greater than the rest mass of the electron (m_e=0.511MeV). So if we were doing this problem entirely correctly we would have used the relativistic energy relation $E^2=p^2c^2+m_e^2c^4$.

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(* 2. We know that $T=\left[1+\frac{\sin^2(k_2 a)}{4E/V_0(E/V_0-1)}\right]^{-1}$ and R= 1-T. We must take care to use the correct value of a. In this formula a refers to the entire width of the well thus here a= 2nm. From the given energy we get a value of k_2= $7.24\times10^{9}\text{m}^{-1}$. This leads to R=0.228.

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(* 3. For $E>U_0$, the reflection amplitude is $|r|^2=R=\frac{\sin^2(k_2 a)}{\sin^2(k_2 a)+4\frac{k_2^2 k_1^2}{(k_2^2-k_1^2)^2}}$,

where $k_2=\frac{\sqrt{2m(E-U_0)}}{\hbar}$, $k_1=\frac{\sqrt{2mE}}{\hbar}$. k_2 and k_1 are both real, hence $\frac{k_2^2 k_1^2}{(k_2^2-k_1^2)^2}>0$, the denominator is larger than the numerator. Therefore $|r|\leq1$;

For $E<U_0$, k_2 is imaginary. The reflection amplitude is $|r|^2$=

$$R=\frac{\sinh^2\left(\frac{\sqrt{2m(U_0-E)}}{\hbar}a\right)}{\sinh^2\left(\frac{\sqrt{2m(U_0-E)}}{\hbar}a\right)+4\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)}$$

. Now $\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)>0$,

hence the denominator is larger than the numerator. Therefore $|r|\leq1$. Also you can note that R= $1-T=1-\left[1+\frac{\sin^2(ka)}{4E/V_0(E/V_0-1)}\right]^{-1}$. Since T is always zero or a positive number, R is a number which is smaller than or equal to 1. We know that $R=|r|^2$ thus meaning that $|r|\leq1$. If r were a number greater than 1, that would mean that particles were being created at the boundary of the potential which can not be true and we know that a value of r less than 0 has no physical meaning.

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(* 4. (a)

$$|\psi_I|^2=|Ae^{ikx}+Be^{-ikx}|^2=|A|^2\left|e^{ikx}-\frac{\alpha+ik}{\alpha-ik}e^{-ikx}\right|^2=|A|^2\left(e^{ikx}-\frac{\alpha+ik}{\alpha-ik}e^{-ikx}\right)\left(e^{-ikx}-\frac{\alpha-ik}{\alpha+ik}e^{ikx}\right)$$

$$=|A|^2\left(1-\frac{\alpha+ik}{\alpha-ik}e^{-2ikx}-\frac{\alpha-ik}{\alpha+ik}e^{2ikx}+1\right)=|A|^2\frac{2(\alpha^2+k^2)-(\alpha+ik)^2e^{-2ikx}-(\alpha-ik)^2e^{2ikx}}{\alpha^2+k^2}$$

$$=|A|^2\frac{2(\alpha^2+k^2)-(\alpha^2-k^2)(e^{-2ikx}+e^{2ikx})+2ik\alpha(e^{2ikx}+e^{-2ikx})}{\alpha^2+k^2}$$

$$=|A|^2\frac{2(\alpha^2+k^2)-2(\alpha^2-k^2)(\cos^2(kx)-\sin^2(kx))-8\sin(kx)\cos(kx)}{\alpha^2+k^2}$$

$$=4|A|^2\left(\frac{\alpha}{\sqrt{\alpha^2+k^2}}\sin(kx)-\frac{k}{\sqrt{\alpha^2+k^2}}\cos(kx)\right)^2=4|A|^2(\sin(kx)\cos\theta-\cos(kx)\sin\theta)^2,$$

where $\tan\theta=\frac{k}{\alpha}$. Thus $|\psi_I|^2=4|A|^2\sin^2(kx-\theta)$.

(b) If k=0,

then $\theta=0°$ and D=0. The wave is zero at the step. The step is relatively so high that the wave does not penerate it.

If $\alpha=0$, then $\theta=90°$ and D=2A. The wave is maximum at the step and there is much penetration.

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(* 5. This is not tunneling;

the kinetic energy is never negative and the wave function between 0 and L is thus of the form $e^{ik'x}$ not $e^{-\alpha x}$. Therefore, we need only replace U_0 by $-U_0$ in the potential barrier reflection equation (6-13).

$$R=\frac{\sin^2\left(\frac{\sqrt{2m(E+U_0)}}{\hbar}L\right)}{\sin^2\left(\frac{\sqrt{2m(E+U_0)}}{\hbar}L\right)+4\frac{E}{U_0}\left(\frac{E}{U_0}+1\right)}.$$

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