Problem Set 5

1. The electron in an infinite quantum well with quantum number n has energy

$E_n = \frac{\hbar^2 \pi^2 n^2}{2 m a^2}$. Hence when it makes a transition from the state n to the ground state,

the energy released is $\Delta E = E_n - E_1 = \frac{\hbar^2 \pi^2}{2 m a^2} (n^2 - 1) = \frac{h c}{\lambda}$,

where λ is the wavelength of the emitted photon.

Therefore the wavelength of the photon is $\lambda = \frac{2 h c m a^2}{\hbar^2 \pi^2 (n^2 - 1)}$.

2. If both n and m are odd numbers :

$$\int_{-a/2}^{a/2} \psi_n^* (x)\, \psi_m (x)\, dx =$$

$$\frac{2}{a} \int_{-a/2}^{a/2} \cos\left(\frac{n \pi}{a} x\right) \cos\left(\frac{m \pi}{a} x\right) dx = \text{(* change of variable: } z = \tfrac{\pi}{a} x \text{ *)}$$

$$\frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos(n z) \cos(m z)\, dz = \text{(* product of two even functions is still an even function *)}\ \frac{4}{\pi} \int_0^{\pi/2} \cos(n z) \cos(m z)\, dz =$$

$$\text{(* } n \neq m \text{ *)}\ \frac{4}{\pi} \frac{1}{n^2 - m^2} \left(n \sin\left(\frac{n \pi}{2}\right) \cos\left(\frac{m \pi}{2}\right) - m \cos\left(\frac{n \pi}{2}\right) \sin\left(\frac{m \pi}{2}\right)\right)$$

Since n and m are odd integers, $\cos\left(\frac{n \pi}{2}\right)$ and $\cos\left(\frac{m \pi}{2}\right)$ are

both 0. Then we get $\int_{-a/2}^{a/2} \psi_n^* (x)\, \psi_m (x)\, dx = 0$;

If both n and m are even numbers :

$$\int_{-a/2}^{a/2} \psi_n^* (x)\, \psi_m (x)\, dx =$$

$$\frac{2}{a} \int_{-a/2}^{a/2} \sin\left(\frac{n \pi}{a} x\right) \sin\left(\frac{m \pi}{a} x\right) dx = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \sin(n z) \sin(m z)\, dz = \text{(* product of two odd functions is an even function *)}\ \frac{4}{\pi} \int_0^{\pi/2} \sin(n z) \sin(m z)\, dz =$$

$$\frac{4}{\pi} \frac{1}{n^2 - m^2} \left(m \sin\left(\frac{n \pi}{2}\right) \cos\left(\frac{m \pi}{2}\right) - n \cos\left(\frac{n \pi}{2}\right) \sin\left(\frac{m \pi}{2}\right)\right)$$

Since n and m are even integers, $\sin\left(\frac{n \pi}{2}\right)$ and $\sin\left(\frac{m \pi}{2}\right)$ are

both 0. Then we get $\int_{-a/2}^{a/2} \psi_n^* (x)\, \psi_m (x)\, dx = 0$;

If n is an odd number and m is an even number :

$$\int_{-a/2}^{a/2} \psi_n^*(x)\,\psi_m(x)\,dx = \frac{2}{a}\int_{-a/2}^{a/2} \cos\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx = 0$$

(* the product of an odd function and an even funciton is an odd funciton. And the integration of an odd function over a range symmetric about origin is 0. *)

Hence for any n ≠ m, the normalized eigenfunctions of the infinite potential well have the property $\int_{-a/2}^{a/2} \psi_n^*(x)\,\psi_m(x)\,dx = 0.$

3. (a)

Since ψ is properly normalized, we have

$$\int_{-a/2}^{a/2} \psi^*(x)\,\psi(x)\,dx = 1.$$

Plug the expansion of ψ in, and we find :

$$\int_{-a/2}^{a/2} \psi^*(x)\,\psi(x)\,dx =$$

$$1 = \int_{-a/2}^{a/2} (A\,\psi_1(x) + B\,\psi_2(x))^*\,(A\,\psi_1(x) + B\,\psi_2(x))\,dx =$$ **(* expand the product. For convenience, drop the x argument in function ψ. *)**

$$\int_{-a/2}^{a/2} |A|^2\,\psi_1^*\,\psi_1 + |B|^2\,\psi_2^*\,\psi_2 + (A^*\,B\,\psi_1^*\,\psi_2 + B^*\,A\,\psi_2^*\,\psi_1)\,dx =$$

$$|A|^2\,\delta_{1,1} + |B|^2\,\delta_{2,2} + (A^*\,B + B^*\,A)\,\delta_{1,2} = |A|^2 + |B|^2.$$

Therefore, $|A|^2 + |B|^2 = 1.$

(b)

Eigenvalues of the infinite square well is $E_n = \frac{\hbar^2\pi^2 n^2}{2 m a^2}$**. Hence the probability of getting energy** $E_1 = \frac{\hbar^2\pi^2}{2 m a^2}$ **for an energy measurement is** $|A|^2$**,**

and the probability of getting $E_2 = \frac{2\hbar^2\pi^2}{m a^2}$ **is** $|B|^2$**;**

The expectation value of the energy for this wave function is therefore $\langle \hat{H} \rangle = |A|^2 \frac{\hbar^2\pi^2}{2 m a^2} + |B|^2 \frac{2\hbar^2\pi^2}{m a^2}.$

4. $\psi_2 = \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi}{a}x\right),\ E_2 = \frac{2\hbar^2\pi^2}{m a^2};$

$\psi_3 = \sqrt{\frac{2}{a}}\cos\left(\frac{3\pi}{a}x\right),\ E_3 = \frac{9\hbar^2\pi^2}{2 m a^2};$

Hence the probability that the electron is in the domain $\left[-\frac{a}{2}, 0\right]$ **is :**

In[1]:=
$$\int_{-\frac{a}{2}}^{0}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}}\left(\mathrm{Sin}\left[\frac{2\pi}{a}x\right]E^{i\frac{2\hbar\pi^2}{m a^2}t}+\mathrm{Cos}\left[\frac{3\pi}{a}x\right]E^{i\frac{9\hbar\pi^2}{2 m a^2}t}\right)\right)\left(\frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}}\left(\mathrm{Sin}\left[\frac{2\pi}{a}x\right]E^{-i\frac{2\hbar\pi^2}{m a^2}t}+\mathrm{Cos}\left[\frac{3\pi}{a}x\right]E^{-i\frac{9\hbar\pi^2}{2 m a^2}t}\right)\right)dx$$

Out[1]=
$$\frac{1}{2}+\frac{4\,\mathrm{Cos}\left[\frac{5\pi^2 t\hbar}{2a^2 m}\right]}{5\pi}$$

The period of oscillation of this probability is $\tau = 2\frac{\pi}{\omega} = \frac{4a^2 m}{5\pi\hbar}$.