Problem set 4

1. (a)

$$\int_{-\infty}^{\infty}\left(\frac{C}{1+\left(\frac{x}{a}\right)^2}\right)^*\left(\frac{C}{1+\left(\frac{x}{a}\right)^2}\right)dx =$$

$$\int_{-\infty}^{\infty}\frac{C^2}{\left(1+\left(\frac{x}{a}\right)^2\right)^2}dx = a\,C^2\int_{-\infty}^{\infty}\frac{1}{\left(1+\left(\frac{x}{a}\right)^2\right)^2}d\frac{x}{a} = a\,C^2\int_{-\infty}^{\infty}\frac{1}{(1+u^2)^2}du = a\,C^2\,\frac{\pi}{2} = 1$$

Hence we get $C\,(a) = \sqrt{\frac{2}{a\,\pi}}$, the dimension of C is $m^{-\frac{1}{2}}$.

(b)

In[11]:= $\text{psi}[x\_, a\_] := \frac{\sqrt{\frac{2}{a\,\pi}}}{1+\left(\frac{x}{a}\right)^2};$

```
psi[1, 2.5]
```

Out[12]= 0.435023

(c) For a normalized wavefunction,
the integration of the probability density over all space is 1.
Check normalization of ψ for a = 2.5, 4, and 6 :

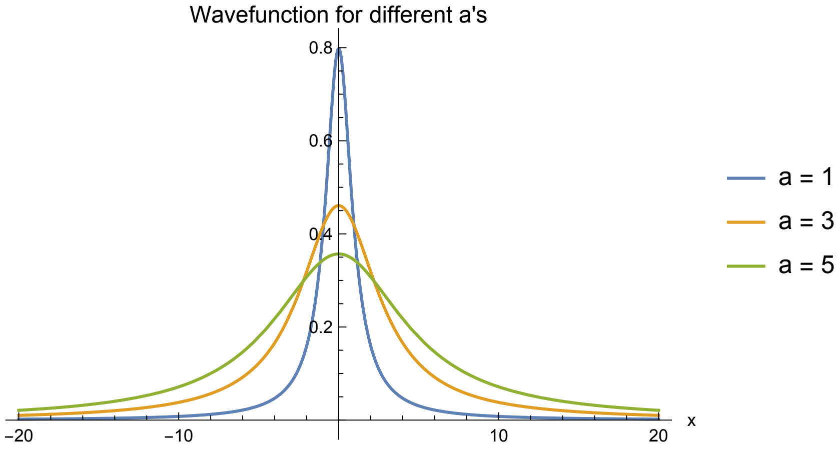
In[13]:=
```
Integrate[Conjugate[psi[x, #]] * psi[x, #] & /@ {2.5, 4., 6.},
  {x, -Infinity, Infinity}]
(* here the Pure function is used
  (if you are familiar with Matlab, the pure function in
     Mathematica is similar to the anonymous function in Matlab). When
  using pure function, '#' is the variable that to be mapped
  onto the list starting with '/@',
and the funciton itself always ends with '&'. Refer to
  the documentation for more details and examples. *)
```

Out[13]= {1., 1., 1.}

(d)

In[14]:=
```
Plot[{psi[x, 1.], psi[x, 3.], psi[x, 5.]},
 {x, -20, 20}, PlotLegends → {"a = 1", "a = 3", "a = 5"},
 PlotLabel → "Wavefunction for different a's",
 AxesLabel → {"x"}, PlotRange → Full]
```

Out[14]=



For smaller a, we get a wavefunciton with samller broadening, which means the wave is more localized.

2. (a)

Do the integration by hand :

$$< \hat{x} > = \int_{-\infty}^{\infty} \sqrt{\frac{2}{a\pi}} \left(\frac{1}{1+\left(\frac{x}{a}\right)^2}\right)^* x \sqrt{\frac{2}{a\pi}} \left(\frac{1}{1+\left(\frac{x}{a}\right)^2}\right) dx = \frac{2}{a\pi} \int_{-\infty}^{\infty} x \left(\frac{1}{1+\left(\frac{x}{a}\right)^2}\right)^2 dx =$$

$$\frac{a}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{1+\left(\frac{x}{a}\right)^2}\right)^2 d\frac{x^2}{a^2} = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{1}{(1+u)^2} du = \frac{a}{\pi} \left(-\frac{1}{1+u}\right)_{-\infty}^{\infty} = 0.$$

Use Mmathematica to do the integration :

In[15]:=
```
Integrate[Conjugate[psi[x, a]] * x * psi[x, a], {x, -∞, ∞}]
```

Out[15]= 0

(b)

Do the integration by hand :

$$< \hat{x^2} > = \int_{-\infty}^{\infty} \sqrt{\frac{2}{a\pi}} \left(\frac{1}{1+\left(\frac{x}{a}\right)^2}\right)^* x^2 \sqrt{\frac{2}{a\pi}} \left(\frac{1}{1+\left(\frac{x}{a}\right)^2}\right) dx =$$

$$\frac{2}{a\pi} \int_{-\infty}^{\infty} x^2 \left(\frac{1}{1+\left(\frac{x}{a}\right)^2}\right)^2 dx = \frac{2a^2}{\pi} \int_{-\infty}^{\infty} \left(\frac{x}{a}\right)^2 \left(\frac{1}{1+\left(\frac{x}{a}\right)^2}\right)^2 d\frac{x}{a} = \frac{2a^2}{\pi} \frac{\pi}{2} = a^2$$

Use Mmathematica to do the integration :

In[16]:=
```
Integrate[Conjugate[psi[x, a]] * x^2 * psi[x, a], {x, -∞, ∞}, Assumptions → a > 0]
```

Out[16]= a^2

3. (a)

$\hat{x}\,\psi(x) = \hat{x}\cos(kx) = x\cos(kx)$, which can not be written in the form of a constant times ψ. Hence ψ is not an eigenfunction of the position operator.

(b)

$$\hat{p}\,\psi(x) = -i\hbar \frac{d}{dx}\cos(kx) = i\hbar k \sin(kx),$$

which can not be written in the form of a constant times ψ. Hence ψ is not an eigenfunction of the momentum operator.

(c)

$$\hat{H}\,\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\cos(kx) = \frac{\hbar^2 k^2}{2m}\cos(kx),$$

where $\frac{\hbar^2 k^2}{2m}$ is a constant. Hence ψ is an eigenfunction of the Hamiltonian, and the eigenvallue is $\frac{\hbar^2 k^2}{2m}$. The wavefunction will evolve as

$$e^{-i\frac{\hbar k^2}{2m}t}\cos(kx).$$