Physics 217
Problem Set 7
Due: Fri, October 26th, 2018

Consider a particle in the ground state of an infinite well of width a,

$$\begin{aligned} \psi(x) &= \sqrt{\tfrac{2}{a}}\cos(\pi x/a) \quad (-a/2 < x < a/2) \\ &= 0 \quad\quad\quad\quad\quad\quad\; (x < -a/2,\ x > a/2) \end{aligned}$$

At time $t = 0$ we suddenly take away the "walls" of the well, setting $V(x) = 0$ everywhere. The state is still $\psi(x)$, but it is now appropriate to write this state as a superposition of plane waves,

$$\psi(x) = \int_{-\infty}^{\infty} \tilde{\psi}(k) \frac{\exp(ikx)}{\sqrt{2\pi}} \, dk \ . \qquad (*)$$

1. Calculate $\tilde{\psi}(k)$, the Fourier transform of $\psi(x)$. (Remember that $\sin\theta = (\exp(i\theta) - \exp(-i\theta)/(2i)$.) Calculate $|\tilde{\psi}(k)|^2$. Check that $|\tilde{\psi}(k)|^2$ is normalized correctly: the easiest way is to evaluate the integral numerically (using Mathematica or other software) for various values of a. Make a plot of $|\tilde{\psi}(k)|^2$ for k from -0.2 to 0.2, for $a = 100$.

2. Calculate k_0, the lowest value of k at which $|\tilde{\psi}(k)|^2 = 0$. As the infinite well becomes wider (increasing a), what happens to k_0? Note from your plot of $|\tilde{\psi}(k)|^2$ that the dominant contribution to $\tilde{\psi}$ comes from $0 < k < k_0$, so k_0 is an estimate of Δk, the range of wavenumbers of the plane waves that constitute $\psi(x)$. The uncertainty in position is $\Delta x = a$. Write down $\Delta x \Delta k$ and hence $\Delta x \Delta p$. Is your result consistent with Heisenberg's uncertainty relation?

3. Equation (*) above shows how, if you add together plane waves of *all* wavenumbers k with the appropriate weights, you can build any function you want to. In this question we will use Mathematica to see how the function becomes closer to what we wanted as we include more wavenumbers.

   First, create a function `psi$tilde(k)` that evaluates your expression for $\tilde{\psi}(k)$ from question 1, for the case where $a = 100$. Now create another function `psi$approx(x)` that makes a crude approximation to the integral in equation (*) by just using the value of $\tilde{\psi}(k)$ at $k = 0.03$. Plot the real part of this function for the range $x = -400$ to $x = 400$. You see that you get a cos wave with a bump as desired between $x = -50$ and $x = 50$, but also bumps elsewhere. *What determined the wavelength of the sine wave?* Improve things by including the contribution from $k = 0.06$. Plot this version too, and see that it is closer to what we wanted.

To continue this by including the contributions from different wavenumbers, modify your function `psi$approx(x)` so that it takes both x and k as its arguments. Then add this line to your code:

```
plot$approx[start_, end_, incre_] :=
  Plot[Re[Sum[psi$approx[x, i], {i, start, end, incre}]],
     {x, -400, 400}, PlotRange -> All, AxesLabel -> {"x"},
   PlotLabel -> "Approximated wavefunciton psi(x)"];
```

The function `plot$approx` sums the comtributions of `psi$approx(x, k)` for the k's from *start* to *end*, incremented by *incre*, and then plot the approximated function. Reproduce your previous result with this function.

Add more wavenumbers, see how the sum of plane waves approximates the desired wavepacket more and more closely as we allow contributions from plane waves with a wider range of wavelengths.

Why is it wavenumbers k of order 0.05 that are important? What goes wrong if you leave out the lower wavenumbers? What goes wrong if you leave out the higher wavenumbers?