Problem set 10

(* 1.

$$\Phi^*_{m_l}(\phi)\,\Phi_{m_l}(\phi) = \left(A^* e^{-im_l\phi} + B^* e^{+im_l\phi}\right)\left(Ae^{+im_l\phi} + Be^{im_l\phi}\right)$$
$$= |A|^2 + |B|^2 + A^* B e^{-2imm_l\phi} + B^* A e^{+2im_l\phi};$$

The last two terms are manifestly functions of ϕ. The only way they can be eliminated is if either A or B is zero. (A and B can not both be zero, or the solution itself would be zero.) Thus, we are left with either $Ae^{+im_l\phi}$ or $Be^{im_l\phi}$, which, if m_l may take on both positive and negative integer values, are equivalent.

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(* 2.

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)Ae^{-br} = \frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(-bAr^2 e^{-br}\right)$$
$$= \frac{-\hbar^2}{2m}\frac{1}{r^2}\left(b^2 r^2 - 2br\right)Ae^{-br}$$
$$= \frac{-\hbar^2 b^2}{2m}Ae^{-br} + \frac{\hbar^2 b}{m}\frac{1}{r}Ae^{-br};$$

The three other terms in radial equation (7-31) are proportional to $\frac{1}{r^2}Ae^{-br}$, $\frac{1}{r}Ae^{-br}$ and Ae^{-br}. Neither of the two terms we have thus far could cancel the first, so its coefficient must be zero, implying that l= 0. The term $\frac{\hbar^2 b}{m}\frac{1}{r}Ae^{-br}$ must cancel the potential energy term in (7-31).

$$\frac{\hbar^2 b}{m}\frac{1}{r}Ae^{-br} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}Ae^{-br} \implies b = \frac{me^2}{4\pi\epsilon_0\hbar^2}.$$ Finally,

the remaining terms must be equal: $\frac{-\hbar^2 b^2}{2m}Ae^{-br} = EAe^{-br} \implies E = -\frac{\hbar^2 b^2}{2m} = -\frac{\hbar^2\left(\frac{me^2}{4\pi\epsilon_0\hbar^2}\right)^2}{2m}$

$$= -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2}.$$ This is the correct ground-state energy.

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(* 3.

$$[\mathbb{L}^2, L_z] = \left[L_x^2 + L_y^2 + L_z^2,\ L_z\right] = [L_x^2,\ L_z] + \left[L_y^2,\ L_z\right] + [L_z^2,\ L_z]$$
$$= (L_x L_x L_z - L_z L_x L_x) + \left(L_y L_y L_z - L_z L_y L_y\right) + 0$$
$$= L_x(L_x L_z - L_z L_x) + (L_x L_z - L_z L_x)L_x + L_y\left(L_y L_z - L_z L_y\right) + \left(L_y L_z - L_z L_y\right)L_y$$
$$= L_x[L_x,\ L_z] + [L_x,\ L_z]L_x + L_y\left[L_y,\ L_z\right] + \left[L_y,\ L_z\right]L_y$$
$$= i\hbar\left(-L_x L_y + -L_y L_x + L_y L_x + L_x L_y\right)$$
$$= 0.$$

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(* 4.
$Y_{1,1}(\theta,\ \phi)=$
$-\sqrt{\frac{3}{8\pi}}\ (\sin\theta)e^{i\phi}$. Plug it into the angular part of the Schrodinger equation:

$$\frac{d^2}{d\theta^2}Y_{1,1}+\cot\theta\frac{d}{d\theta}Y_{1,1}+\frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2}Y_{1,1}=-\sqrt{\frac{3}{8\pi}}\ \left(-\sin\theta\ e^{i\phi}+\cot\theta\ \cos\theta\ e^{i\phi}+\frac{1}{\sin^2\theta}\sin\theta\ \left(-e^{i\phi}\right)\right)$$

$$=-\sqrt{\frac{3}{8\pi}}\ \frac{e^{i\phi}}{\sin\theta}(-\sin^2\theta+\cos^2\theta-1)$$

$$=-\sqrt{\frac{3}{8\pi}}\ \frac{e^{i\phi}}{\sin\theta}(-2\sin^2\theta)$$

$$=2\sqrt{\frac{3}{8\pi}}\ e^{i\phi}\sin\theta\ =-1(1+1)Y_{1,1}(\theta,\ \phi).$$

$$\frac{d^2}{d\phi^2}Y_{1,1}=-\sqrt{\frac{3}{8\pi}}\ (\sin\theta)\left(-e^{i\phi}\right)=-1^2Y_{1,1}(\theta,\ \phi).$$

Therefore the l=
1. m_l=1 spherical harmonic $Y_{1,1}(\theta,\ \phi)$ is a solution of the angular part of the Schrodinger equation for a central potential.

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