1. Consider the following statement: If at all times during a projectile's flight its speed is much les than the terminal speed v_{ter}, the effects of the air resistance aew usually very small.

(a) Without reference to the explicit equations for the magnitude of v_{ter}, explain clearly why this is so.

SOLUTION - Without air resistance, a free falling projectile would keep accelerating due to gravity, meaning its speed would only get bigger with time. Because there is air resistance, that doesn't happen: as the speed increases so does the air resistance, stalling the projectile. Terminal speed happens when when gravity is matched by the air resistance effects, thus once the projectile achieves v_{ter} its speed remains constant, ultimately working as an upper limit for speed. If the speed is much lower than this cap, air resistance effects won't be as relevant, as the drag should be much less than the weight.

(b) By examining the explicit equations in the book (2.26) and (2.53) explain why the statement above is even more useful for the case of quadratic drag than for the lienar case. [Hint: Express the ratio f/mg of the drag to the weight in terms of the ratio v/v_{ter}.]

SOLUTION - Writing the ratio of linear and quadratic drag to the weight, we have

$$|f_{linear}| = bv; \qquad |f_{quad}| = cv^2 \tag{1}$$

$$\frac{f_{linear}}{mg} = \frac{bv}{mg} = \frac{v}{v_{ter}}; \qquad \frac{f_{quad}}{mg} = \frac{cv^2}{mg} = \frac{v^2}{v_{ter}^2} \tag{2}$$

From the ratio, we have that a small value v wil result in a more dramatic change in the quadratic drag given its squared dependence on v, making the statement even more useful - for a v that is far from the terminal velocity the quadratic drag corrections would be even smaller.

2. In Fig. 1 a toy is shown, which has the shape of a cylinder mounted on top of a hemishpere. The radius of the hemisphere is R and the center-of-mass (CM) of the whole toy is at a height h above the floor.

(a) Write down the gravitational potential energy when the toy is tipped to an angle θ from the vertical. Note that you will need to find the height of the CM as a function of θ. It may help to think first about the height of the hemisphere's center O as the toy tilts.

SOLUTION - From the picture, we have the following definitions:

$$d = h - R \tag{3}$$

$$y = (h - R)\cos\theta + R \tag{4}$$

$$U = mgy = mgR + mg(h - R)\cos\theta \tag{5}$$

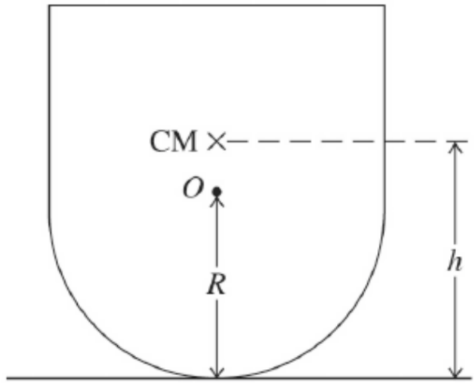

Figure 1: Picture for problem 2

(b) For what values of R and h is the equilibruim at $\theta = 0$ stable?
SOLUTION - To check if a given equilibrium point is stable, we need to make sure the second derivative of U is positive

$$\frac{\partial^2 U}{\partial \theta^2} = -mg(h - R)\cos\theta \tag{6}$$

At $\theta = 0$ we have $\cos\theta = 1$, therefore for this point to be stable $h > R$ $\square$

3. A small cart (mass m) is mounted on rails inside a large cart. The two are attached by a spring (constant k) in such a way that the small cart is in equilibrium at the midpoint of the large one. The distance of the small cart from equilibruim is denoted by x and that of the large one from a fixed point on the ground is X as shown in Fig. 2. The large cart is now forced to oscilate such that $X = A\cos\omega t$, with both A and ω fixed.

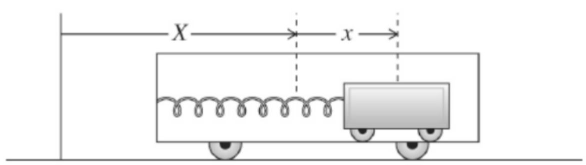

Figure 2: Picture for problem 3

(a) Set up the Lagrangian for the motion of the small cart.

SOLUTION - To determine the Lagrangian, we start by writing the postion of the small cart as a function of time

$$x_{small} = X + x = A\cos(\omega t) + x \tag{7}$$
$$\dot{x_{small}} = -A\omega\sin(\omega t) + \dot{x} \tag{8}$$

So its Lagrangian is given by

$$L = T - U = \frac{1}{2}m(\dot{x}^2 + A^2\omega^2\sin^2(\omega t) - \dot{x}A\omega\sin(\omega t)) - \frac{1}{2}kx^2 \tag{9}$$

(b) Show that the equation of motion (Lagrange equation) has the form

$$\ddot{x} + \omega_0^2 x = B\cos\omega t \tag{10}$$

where ω_0^2 is the natural frequency $\omega_0 = \sqrt{k/m}$ and B is a constant.
SOLUTION - The Euler-Lagrange equation can be calculated as follows:

$$\frac{\partial L}{\partial x} = -kx \tag{11}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} - mA\omega\sin(\omega t) \tag{12}$$

$$m\ddot{x} - mA\omega^2\cos(\omega t) = -kx \tag{13}$$

Now, applying the definitions given by the problem, we can rewrite equation 13 as

$$\ddot{x} + \omega_0^2 x = B\cos(\omega t) \tag{14}$$

where $\omega_0 = \sqrt{k/m}$ and $B = A\omega^2$ $\square$

4. The spherical pendulum is just a simple pendulum that is free to move in any sideways direction under the influence of gravity. The bob of a spherical pendulum moves on a sphere, centered on the point of support with radius $r = R$, the fixed lenght of the pendulum. A convenient choice of coordinates is r, θ, ϕ as indicated in Fig. 3. Choose θ and ϕ as generalized coordinates.

(a) Determine velocity and kinectic energy.
SOLUTION - First we need to write the position of the bob in the required coordinates, by making a transformation from Cartesian to spherical coordinates:

$$x = R\sin\theta\cos\phi \tag{15}$$

$$y = R\sin\theta\sin\phi \tag{16}$$

$$z = R\cos\theta \tag{17}$$

Using these definitions we can write $v^2 = x^2 + y^2 + z^2$:

$$v^2 = (R\dot{\theta}\cos\theta\cos\phi - R\dot{\phi}\sin\theta\sin\phi)^2 + (R\dot{\theta}\cos\theta\sin\phi + R\dot{\phi}\sin\theta\cos\phi)^2 + R^2\dot{\theta}^2\sin^2\theta \tag{18}$$

$$= R^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) \tag{19}$$

Now its trivial to write the kinectic energy,

$$T = \frac{1}{2}mR^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) \tag{20}$$

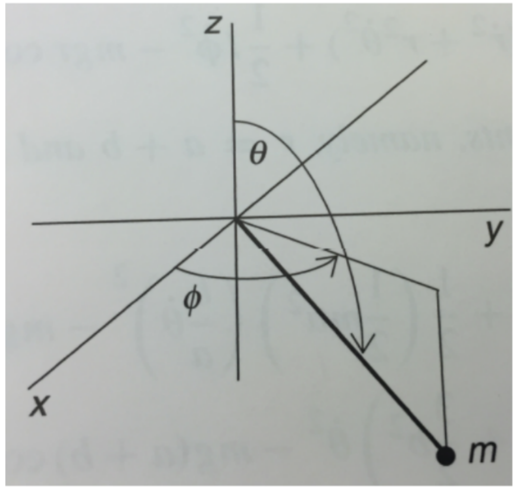

Figure 3: Picture for problem 4

(b) Determine the potential energy and the Lagrangian.

SOLUTION - The potential energy and Lagrangian are given by

$$U = -mgz = -mgR\cos\theta \tag{21}$$

$$L = \frac{1}{2}mR^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgR\cos\theta \tag{22}$$

(c) Determine the two Lagrange equations.

SOLUTION - The equations of motion are

$$\theta : R\ddot{\theta} = R\dot{\phi}^2\sin\theta\cos\theta - g\sin\theta \tag{23}$$

$$\phi : \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\phi}} = 0 \tag{24}$$

(d) What does the ϕ equation tell us about the z component of the angular momentum l_z?

SOLUTION - From equation 24, we see that $mR^2\dot{\phi}\sin^2\theta$ is constant, meaning that the z component of the angular momentum is conserved.

(e) For the special case ϕ =const, describe what the θ equation tells us.

SOLUTION - For this case, the equation of motion for θ reduces to

$$R\ddot{\theta} = -g\sin\theta \tag{25}$$

which is the equation for a simple pendulum confined to a plane, as expected when we force ϕ to be fixed.

(f) Use the ϕ equation to eliminate $\dot{\phi}$ by l_z in the θ equation and discuss the existence of an angle θ_0 at which θ can remain constant. Why is this motion called a conical pendulum?

SOLUTION - Writing $\dot{\phi}$ in terms of l_z, we can rewrite equation 23 as follows

$$\dot{\phi} = \frac{l_z}{mR^2 \sin^2\theta} \tag{26}$$

$$R\ddot{\theta} = \kappa \frac{\cos\theta}{\sin^3\theta} - g\sin\theta, \tag{27}$$

where $\kappa = \frac{l_z^2}{m^2R^3}$. Forcing θ to be constanst means that $\ddot{}\,.\,\theta = 0$, so

$$\kappa\cos\theta = g\sin^4\theta \tag{28}$$

If $\frac{\pi}{2} < \theta < \pi$, $\cos\theta < 0$ and $\sin^4\theta$ and this equation can't be satisfied. However, if $0 < \theta < \frac{\pi}{2}$, $\cos\theta$ goes from 1 to 0 and $\sin\theta$ goes from 0 to 1, which means there is an angle θ_0 that will satify tihs condition. In this particular case, the motion of the string traces out a vertical cone that has an angle equal to $\theta_0/2$, hence the name.