1. Problem 9.15 from Taylor

SOLUTION - We start describing gravity as follows:

$$\mathbf{g} = \mathbf{g_0} + (\boldsymbol{\Omega} \times \mathbf{R}) \times \boldsymbol{\Omega} \tag{1}$$

where the second termon the right hand side is the centrifugal acceleration. In spherical coordinates, we can write $\mathbf{R} = (R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)$ and $\boldsymbol{\Omega} = (0, 0, \Omega)$. Thus we have

$$\boldsymbol{\Omega} \times \mathbf{R} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \boldsymbol{\Omega} \\ R\sin\theta\cos\phi & R\sin\theta\sin\phi & R\cos\theta \end{vmatrix} \tag{2}$$

$$= \Omega R\sin\theta\cos\phi\hat{\mathbf{y}} - \Omega R\sin\theta\sin\phi\hat{\mathbf{x}} \tag{3}$$

and

$$(\boldsymbol{\Omega} \times \mathbf{R}) \times \boldsymbol{\Omega} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -\Omega R\sin\theta\sin\phi & \Omega R\sin\theta\cos\theta & 0 \\ 0 & 0 & \Omega \end{vmatrix} \tag{4}$$

$$= \Omega^2 R\sin\theta(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}) \tag{5}$$

With this expression, the gravity of the planet can be written as

$$\mathbf{g} = \mathbf{g_0} + \Omega^2 R\sin\theta(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}) \tag{6}$$

and its magnitude is

$$|\mathbf{g}| = \sqrt{g_0^2 + \Omega^4 R^2 \sin^2\theta} \tag{7}$$

Now we apply the two conditions given by the problem: The first one, $g(0) = g_0$ is already satisfied by equation 7. Now for $g(90) = \lambda g_0$,

$$\lambda g_0 = \sqrt{g_0^2 + \Omega^4 R^2} \tag{8}$$

$$\lambda^2 g_0^2 = g_0^2 + \Omega^4 R^2 \tag{9}$$

$$(\lambda^2 - 1)g_0 = \Omega^4 R^2 \tag{10}$$

Back to equation 7:

$$g = \sqrt{g_0^2 + (\lambda^2 - 1)g_0^2 \sin^2\theta} \tag{11}$$

$$= g_0\sqrt{1 + \lambda^2\sin^2\theta - \sin^2\theta} \tag{12}$$

$$= g_0\sqrt{\cos^2\theta + \lambda^2\sin^2\theta} \quad \square \tag{13}$$

2. Problem 9.19 from Taylor

(a) **SOLUTION** - For an observer in the inertial frame there are no forces acting on the puck and therefore they will see it moving in a straight line, tangential to the circular motion of the meery go round.
For an observer in the nonintertial frame, te puck is initially at rest and the only force acting on it is the centrifugal force. After some time, the puck will have some velocity, meaning that there will be also a Corriolis force acnting on it. The Corriolis force is always perpendicular to the velocity and deflects the puck to the right, as the centrifugal force is always radial, creating a outward spiral path.

(b) **SOLUTION** - For an observer in the inertial frame, again there are no forces acting on the puck so after it hits the merry go round it remains stationary.
For an observer in the nonintertial frame, the puck is moving in a circle with angular velocity equals to $-\Omega$ as a result of the forces acting on it:

$$\begin{aligned}
\mathbf{F}_{Corriolis} &= 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega} &(14)\\
&= 2mr(-\Omega\hat{\theta}) \times \Omega\hat{\mathbf{z}} &(15)\\
&= -2mr\Omega^2\hat{\mathbf{r}} &(16)\\
\mathbf{F}_{centrifugal} &= m\Omega r(\hat{\mathbf{x}} \times \hat{\mathbf{r}}) \times \Omega\hat{\mathbf{z}} &(17)\\
&= mr\Omega^2\hat{\mathbf{r}} &(18)\\
\mathbf{F}_{total} &= -mr\Omega^2\hat{\mathbf{r}} &(19)
\end{aligned}$$

3. Problem 9.26 from Taylor

SOLUTION - Starting from the equation 9.53 and setting $\Omega = 0$, we have

$$\begin{aligned}
\dot{x} &= v_{ox} &(20)\\
\dot{y} &= v_{oy} &(21)\\
\dot{z} &= v_{oz} - gt &(22)
\end{aligned}$$

Now back to equation 9.53 we find the equation of motion for each component:

$$\begin{aligned}
\ddot{x} &= 2\Omega\left[v_{oy}\cos\theta - (v_{oz} - gt)\sin\theta\right] &(23)\\
\dot{x} &= 2\Omega t\left[v_{oy}\cos\theta - v_{oz}\sin\theta + \frac{1}{2}gt^2\right] + v_{ox} &(24)\\
x &= \Omega t^2(v_{oy}\cos\theta - v_{oz}\sin\theta) + \frac{1}{3}\Omega g t^3\sin\theta + v_{ox}t &(25)
\end{aligned}$$

$$\begin{aligned}
\ddot{y} &= -2\Omega v_{ox}\cos\theta &(26)\\
\dot{y} &= -2\Omega v_{ox}t\cos\theta &(27)\\
y &== \Omega v_{ox}t^2\cos\theta + v_{oy} &(28)
\end{aligned}$$

$$\ddot{z} = -g + 2\Omega v_{ox} \sin\theta \tag{29}$$
$$\dot{z} = -gt + 2\Omega t v_{ox} \sin\theta \tag{30}$$
$$z = -\frac{1}{2}gt^2 + \Omega t^2 v_{ox} \sin\theta + v_{oz} \quad \square \tag{31}$$

4. Problem 10.6 from Taylor

SOLUTION - Given its symmetry, we know that $X_{CM} = Y_{CM} = 0$. So we only need to calculate Z_{CM}:

$$Z_{CM} = \frac{\int z \mathrm{d}V}{V} \tag{32}$$

Calculating the integral we have

$$Z = \int_a^b \int_0^{\pi/2} \int_0^{2\pi} (r\cos\theta) r^2 \sin\theta \mathrm{d}\phi \mathrm{d}\theta \mathrm{d}r \tag{33}$$
$$= \int_a^b r^3 \mathrm{d}r \int_0^{\pi/2} \cos\theta \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \tag{34}$$
$$= (2\pi)\left(\frac{1}{2}\right)\int_a^b r^3 \mathrm{d}r \tag{35}$$
$$= \pi\left(\frac{1}{4}r^4\Big|_a^b\right) = \frac{\pi}{4}(b^4 - a^4) \tag{36}$$

Since the volume of a hemispherical shell is $\frac{2\pi}{3}(b^3 - a^3)$, from equation 32 we have our answer

$$z_{CM} = \frac{\frac{\pi}{4}(b^4 - a^4)}{\frac{2\pi}{3}(b^3 - a^3)} = \frac{3}{8}\frac{(b^4 - a^4)}{(b^3 - a^3)}. \tag{37}$$

SOLUTION - For the case where a = 0, we have a solid hemisphere and from equation 37 it's straightforward to find Z_{CM}:

$$Z_{CM} = \frac{3b}{8} \tag{38}$$

SOLUTION - For the case where b approaches a, we need to calculate the following limit using the L'opital rule

$$\lim_{b \to a} \frac{b^4 - a^4}{b^3 - a^3} = \frac{4b}{3} \tag{39}$$

Therefore we have

$$Z_{CM} = \frac{3}{8}\left(\frac{4b}{3}\right) = \frac{b}{2} \quad \square \tag{40}$$