1. Problem 9.3 from Taylor

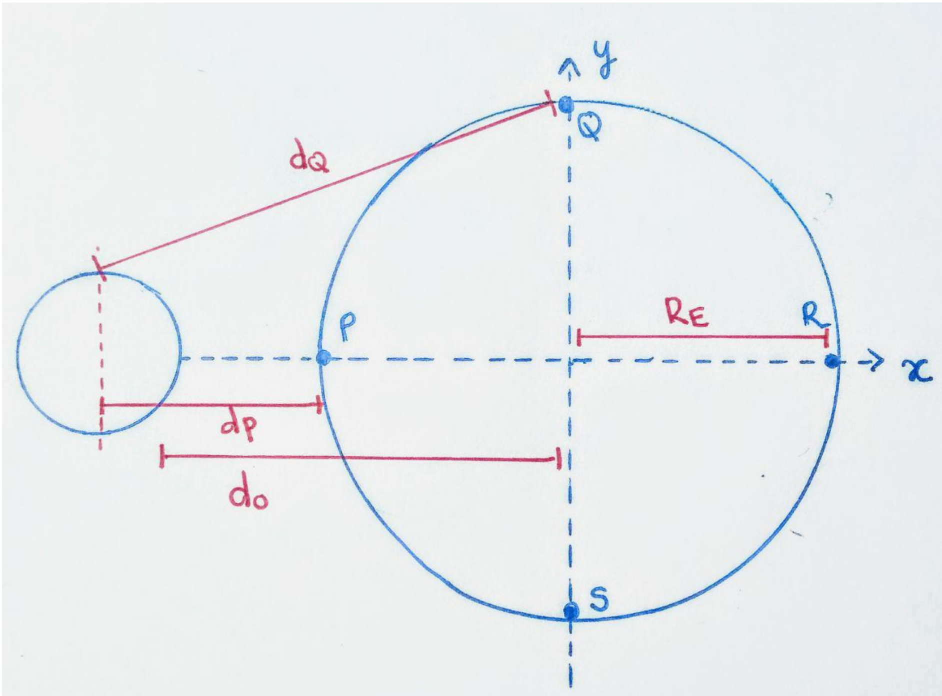

Figure 1: Problem 1 and 2

(a) SOLUTION - From the definition of tidal forces,

$$\mathcal{F}_{tidal} = -GM_m m \left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2} \right) \tag{1}$$

we calculate its magnitude at the point P rewriting the distance d as $d = d_0 - R_E = d_0(1 - R_E/d_0)$:

$$\mathcal{F}_{tidal} = -GM_m m \left(\frac{\hat{d}}{d_0^2} \frac{1}{(1 - R_E/d_0)^2} - \frac{\hat{d}_0}{d_0^2} \right) \tag{2}$$

$$\approx \frac{-GM_m m}{d_0^2} \left[\hat{x} \left(1 + \frac{2R_E}{d_0} \right) - \hat{x} \right] \tag{3}$$

$$= \frac{-GM_m m}{d_0^2} \left(\frac{2R_E}{d_0} \hat{x} \right) = \frac{-2GM_m m R_E}{d_0^3} \hat{x} \tag{4}$$

Numerical comparison with gravitational force:

$$\frac{|\mathcal{F}|}{mg} = \frac{2GM_m R_E}{d_0^3} = 1.1 \times 10^{-7} \tag{5}$$

(b) SOLUTION - Now for point R, we rewrite d as $d = d_0(1 + R_E/d_0)$:

$$\mathcal{F}_{tidal} = -GM_m m \left(\frac{\hat{d}}{d_0^2} \frac{1}{(1 + R_E/d_0)^2} - \frac{\hat{d}_0}{d_0^2} \right) \tag{6}$$
$$\approx \frac{-GM_m m}{d_0^2} \left[\hat{x} - \left(1 - \frac{2R_E}{d_0} \right) - \hat{x} \right] \tag{7}$$
$$= \frac{GM_m m}{d_0^2} \left(\frac{2R_E}{d_0} \hat{x} \right) = \frac{2GM_m m R_E}{d_0^2} \hat{x} \quad \square \tag{8}$$

so it has the same magnitude, but opposite direction.

2. Problem 9.4 from Taylor

SOLUTION - Now we calculate the tidal force at point Q, using the following definition for the distance d:

$$\mathbf{d} = \mathbf{d_0} + R_E \hat{y} \tag{9}$$
$$d^2 = d_0^2 + R_E^2 \tag{10}$$
$$\frac{\hat{d}}{d^2} = \frac{\mathbf{d}}{d^3} \tag{11}$$

$$\mathcal{F}_{tidal} = -GM_m m \left(\frac{\mathbf{d_0} + R_E \hat{y}}{d_0^3} \frac{1}{(1 + R_E^2/d_0^2)^{3/2}} - \frac{\mathbf{d_0}}{d_0^3} \right) \tag{12}$$
$$\approx \frac{-GM_m m}{d_0^3} \left[(\mathbf{d_0} + R_E \hat{y}) \left(1 - \frac{3R_E^2}{2d_0^2} \right) - \mathbf{d_0} \right] \tag{13}$$
$$= \frac{-GM_m m}{d_0^3} \left(R_E \hat{y} - \frac{3R_E^2}{2d_0^2} \mathbf{d_0} - \frac{3R_E^3}{2d_0^2} \hat{y} \right) \tag{14}$$

this last expression can be rewritten as

$$\mathcal{F}_{tidal} \approx \frac{-GM_m m}{d_0^3} R_E \hat{y}, \tag{15}$$

because $d_0 \gg R_E$. For the numerical comparison we have

$$\frac{|\mathcal{F}|}{mg} = 5.7 \times 10^{-6} \quad \square \tag{16}$$

3. Problem 9.10 from Taylor

SOLUTION - For any vector $\mathbf{Q}$ we have that

$$\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} \right)_{So} = \left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} \right)_{S} + \mathbf{\Omega} \times \mathbf{Q} \tag{17}$$

Therefore, for a second derivative in time, we can write

$$\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{So} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{So}\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{So} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{So}\left[\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S} + \boldsymbol{\Omega}\times\mathbf{r}\right] \tag{18}$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S}\left[\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S} + \boldsymbol{\Omega}\times\mathbf{r}\right] + \boldsymbol{\Omega}\times\left[\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S} + \boldsymbol{\Omega}\times\mathbf{r}\right] \tag{19}$$

For clarity, let's define $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}_S \equiv \dot{\mathbf{r}}$:

$$\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{So} = \left(\frac{\mathrm{d}\dot{\mathbf{r}}}{\mathrm{d}t}\right)_{S} + \left(\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right)_{S}\times\mathbf{r} + \boldsymbol{\Omega}\times\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S} + \boldsymbol{\Omega}\times\dot{\mathbf{r}} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) \tag{20}$$

$$= \ddot{\mathbf{r}} + \dot{\boldsymbol{\Omega}}\times\mathbf{r} + \boldsymbol{\Omega}\times\dot{\mathbf{r}} + \boldsymbol{\Omega}\times\dot{\mathbf{r}} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) \tag{21}$$

Plugging this into $m\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{So} = \mathbf{F}$, we get

$$m\ddot{\mathbf{r}} + m(\dot{\boldsymbol{\Omega}}\times\mathbf{r}) + 2m(\boldsymbol{\Omega}\times\dot{\mathbf{r}}) + m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) = \mathbf{F} \tag{22}$$

$$m\ddot{\mathbf{r}} = \mathbf{F} + m\underbrace{\mathbf{r}\times\dot{\boldsymbol{\Omega}}}_{\text{azimutal}} + 2m\underbrace{\dot{\mathbf{r}}\times\boldsymbol{\Omega}}_{\text{Corriolis}} + m\underbrace{(\boldsymbol{\Omega}\times\mathbf{r})\times\boldsymbol{\Omega}}_{\text{centrifugal}} \quad \square \tag{23}$$

4. Problem 9.11 from Taylor

SOLUTION - We start writing the Lagragian in the noninertial frame:

$$\mathcal{L} = \frac{1}{2}m(\dot{\mathbf{r}} + \boldsymbol{\Omega}\times\mathbf{r})^2 + U(\mathbf{r}) \tag{24}$$

$$= \frac{1}{2}m\left[\dot{\mathbf{r}}^2 + (\boldsymbol{\Omega}\times\mathbf{r})^2 + 2\dot{\mathbf{r}}(\boldsymbol{\Omega}\times\mathbf{r})\right] \tag{25}$$

Note that $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$. For the x component, we calculate both sides of the Euler-Lagrange equation,

$$\frac{\partial\mathcal{L}}{\partial x} = m(\boldsymbol{\Omega}\times\mathbf{r})\left[\boldsymbol{\Omega}\times\frac{\partial\mathbf{r}}{\partial x}\right] + m\dot{\mathbf{r}}\left[\boldsymbol{\Omega}\times\frac{\partial\mathbf{r}}{\partial x}\right] - \frac{\partial U}{\partial x}. \tag{26}$$

With $\dfrac{\partial\mathbf{r}}{\partial x} = \hat{x}$,

$$\frac{\partial\mathcal{L}}{\partial x} = m(\boldsymbol{\Omega}\times\mathbf{r})(\boldsymbol{\Omega}\times\hat{x}) + m\dot{\mathbf{r}}(\boldsymbol{\Omega}\times\hat{x}) + F_x \tag{27}$$

$$= m\hat{x}[(\boldsymbol{\Omega}\times\mathbf{r})\times\boldsymbol{\Omega}] + m\hat{x}(\dot{\mathbf{r}}\times\boldsymbol{\Omega}) + F_x \tag{28}$$

$$= \{m[(\boldsymbol{\Omega}\times\mathbf{r})\times\boldsymbol{\Omega}] + m(\dot{\mathbf{r}}\times\boldsymbol{\Omega}) + F\}_x \tag{29}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{x}} = m\dot{\mathbf{r}}\hat{x} + m(\boldsymbol{\Omega}\times\mathbf{r})\hat{x} \tag{30}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \{m\ddot{\mathbf{r}} + m(\boldsymbol{\Omega}\times\dot{\mathbf{r}})\}_x \tag{31}$$

So for this component we have

$$\{m\ddot{\mathbf{r}} = m\left[(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega}\right] + 2m(\dot{\mathbf{r}} \times \boldsymbol{\Omega}) + F\}_x \tag{32}$$

And it's easy to see that components y and z will have similar derivations. Putting all together yields

$$m\ddot{\mathbf{r}} = m\left[(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega}\right] + 2m(\dot{\mathbf{r}} \times \boldsymbol{\Omega}) + \mathbf{F} \tag{33}$$

which is identical to equation 9.34 $\square$