

1. (a) Write down the Lagrangian, $L(x_1, x_2, \dot{x}_1, \dot{x}_2)$ for two particles of equal mass, $m_1 = m_2 = m$, confined to the x axis and connected by a spring with potential energy $U = \frac{1}{2}kx^2$. Here x is the extension of the spring, $x = x_1 - x_2 - l$, where l is the spring's natural length.

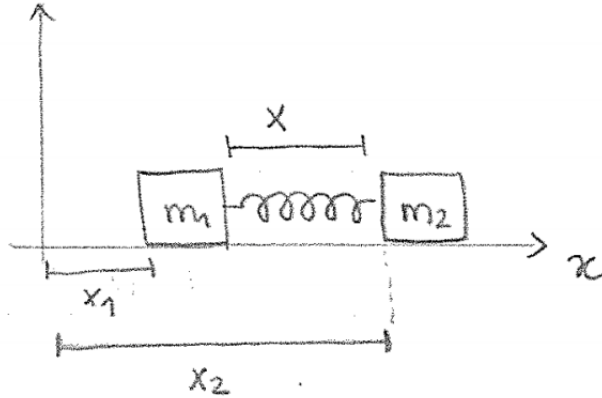


Figure 1: Problem 1

SOLUTION - The Lagrangian for this system is straightforward to calculate given the problem set up:

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 \quad (1)$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}k(x_1 - x_2 - l)^2 \quad (2)$$

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k(x_1 - x_2 - l)^2 \quad (3)$$

- (b) Rewrite the Lagrangian in terms of the center of mass position $X = \frac{x_1 + x_2}{2}$, and the extension x . Write down the two Lagrange equations for X and x and solve for $X(t)$ and $x(t)$. Describe the motion.

SOLUTION - From the definitions of the center of mass position X and the extension x ,

$$x = x_1 - x_2 - l; \quad X = \frac{x_1 + x_2}{2} \quad (4)$$

$$x_1 = X + \frac{x + l}{2}; \quad x_2 = X - \frac{x - l}{2} \quad (5)$$

$$\dot{x}_1 = \dot{X} + \frac{\dot{x}}{2}; \quad \dot{x}_2 = \dot{X} - \frac{\dot{x}}{2} \quad (6)$$

we can rewrite equation 3 as

$$L = \frac{1}{2}m \left[\left(\dot{X} + \frac{\dot{x}}{2} \right)^2 + \left(\dot{X} - \frac{\dot{x}}{2} \right)^2 \right] - \frac{1}{2}kx^2 \quad (7)$$

$$L = m\dot{X}^2 + \frac{1}{4}m\dot{x}^2 - \frac{1}{2}kx^2 \quad (8)$$

The Euler-Lagrange equations for X and x are, respectively:

$$\frac{\partial L}{\partial X} = 0; \quad \frac{\partial L}{\partial \dot{X}} = 2m\dot{X} \quad (9)$$

$$2m\ddot{X} = 0 \quad (10)$$

$$\frac{\partial L}{\partial x} = -kx; \quad \frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m\dot{x} \quad (11)$$

$$\ddot{x} + \frac{k}{m/2}x = 0 \quad (12)$$

From equation 10 it's possible to see that the CM moves as a free particle and the two masses oscillate as a particle of mass $m/2$, identifying ω_0 as $\sqrt{k/(m/2)}$ \square .

2. Consider two particles interacting by a Hooke's law potential energy, $U = \frac{1}{2}kr^2$, where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is their relative position, and subject to no external forces. Show that $\mathbf{r}(t)$ describes an ellipse. Hence, show that both particles move on similar ellipses around their common CM. This is the other instance of an interaction admitting closed orbits.

Hint: Perhaps the easiest way to solve the problem is to use cartesian coordinates, x, y , in the plane of the orbit. Your solution will look like $x = a \cos(\omega t) + b \sin(\omega t)$, and a similar expression for y . You can solve for $\cos(\omega t)$ and $\sin(\omega t)$, and put the equation of the orbit in the form $\alpha x^2 + 2\beta xy + \gamma y^2 = \kappa$, which defines an ellipse if $\alpha > 0$, $\gamma > 0$ and $\alpha\gamma > \beta^2$.

SOLUTION - The Lagrangian for this system has the same form as the previous problem,

$$L = \frac{1}{2}MR^2 + \frac{1}{2}\mu r^2 - \frac{1}{2}kr^2, \quad (13)$$

where M and μ represent the total and reduced mass of the system. Rewriting L in terms of x and y gives

$$L = \frac{1}{2}MR^2 + \frac{1}{2}\mu(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2) \quad (14)$$

and similar equation of motion for both variables:

$$\mu\ddot{x} + kx = 0 \quad \mu\ddot{y} + ky = 0 \quad (15)$$

$$(16)$$

Defining $\omega = \sqrt{k/\mu}$, the solutions for these equations are of the form

$$x = a \cos(\omega t) + b \sin(\omega t); \quad y = c \cos(\omega t) + d \sin(\omega t). \quad (17)$$

Solving for $\cos(\omega t)$ and $\sin(\omega t)$, we find

$$cx - ay = (bc - ad) \sin(\omega t) \quad (18)$$

$$dx - by = (ad - bc) \cos(\omega t) \quad (19)$$

Now we can square and sum these two equations, obtaining

$$(cx - ay)^2 + (dx - by)^2 = (bc - ad)^2 \sin^2(\omega t) + (ad - bc)^2 \cos^2(\omega t) \quad (20)$$

$$\underbrace{(c^2 + d^2)}_{\alpha} x^2 + \underbrace{(a^2 + b^2)}_{\gamma} y^2 + 2 \underbrace{(-ac - db)}_{\beta} xy = \underbrace{(ad - bc)^2}_{\kappa} \quad (21)$$

which defines an ellipse if the conditions for the constants α , γ are positive and $\alpha\gamma > \beta^2$. It's clear that the first two conditions are satisfied as the constants are defined as squared quantities. For the last condition, we can prove it through some algebra:

$$\alpha\gamma = a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2 \quad (22)$$

$$\beta^2 = a^2c^2 + b^2d^2 + 2abcd \quad (23)$$

$$\alpha\gamma - \beta^2 = a^2d^2 + b^2c^2 + 2abcd = (ad + bc)^2 > 0 \quad \square. \quad (24)$$

3. Imagine that the Earth were moving on a circular orbit around the Sun. Show that if the mass of the Sun decreases abruptly to half its value, the Earth's orbit becomes parabolic. You may assume that the reduced mass remains equal to m_E the mass of Earth.

SOLUTION - For a circular orbit, we have that the gravitational force should be equal to the centripetal force,

$$F_{gravity} = F_{centripetal} \quad (25)$$

$$\frac{\gamma}{r^2} = \frac{\mu v^2}{r} \quad (26)$$

$$\mu v^2 = \frac{\gamma}{r} \quad (27)$$

which allows us to write the energy of the system as follows:

$$E = T - U = \frac{1}{2}\mu v^2 - \frac{\gamma}{2} = \frac{\gamma}{2r} - \frac{\gamma}{r} = -\frac{\gamma}{2r} \quad (28)$$

The result is negative, as expected for a circular orbit. When the mass of the sun is halved, the potential energy gets an extra factor of 1/2 as the kinetic energy remains the same:

$$E = \frac{1}{2}\mu v^2 - \frac{\gamma/2}{r} = \frac{\gamma}{2r} - \frac{\gamma}{2r} = 0 \quad (29)$$

which corresponds to a parabolic orbit \square .

4. Suppose that we decide to send a spacecraft to Neptune, using the simple transfer described in Example 8.6 (page 318 in the textbook). The craft starts in a circular orbit close to the earth (radius 1 AU or astronomical unit) and is to end up in a circular orbit near Neptune (radius about 30 AU). Use Kepler's third law to show that the transfer will take about 31 years. (In practice we can do a lot better than this by arranging that the craft gets a gravitational boost as it passes Jupiter).

SOLUTION - Using figure 8.13 from the textbook and assuming an elliptical orbit, the new orbit would have a semi-major axis equals to

$$a_2 = \frac{R_{Earth} + R_{Neptune}}{2} = 15.5AU \quad (30)$$

From Kepler's third law, we can calculate its period directly or by comparison with Earth's values:

$$\frac{\tau_{Earth}}{\tau_{new}} = \left(\frac{a_2}{a_1}\right)^{3/2} \quad (31)$$

$$\tau_{new} = (15.5)^{3/2} \approx 61\text{years} \quad (32)$$

where $\tau_{Earth} = 1\text{year}$ and $a_1 = 1\text{AU}$. The time it takes to change orbits is only half of that, so $t \approx 31\text{years}$ \square