1. A wedge of mass M moves on a horizontal surface. A block of mass m slides down the wedge (see Fig. 7.8 of Taylor's book). Suppose that the wedge has a given motion, $x = \frac{1}{2}at^2$ (a is a fixed constant), **imposed** upon it.

(a) Set up the equations of motion using Newtonian mechanics and determine the constraint force $\mathbf{F_{cstr}}$ between the wedge and the block. Work in the given inertial coordinate system.

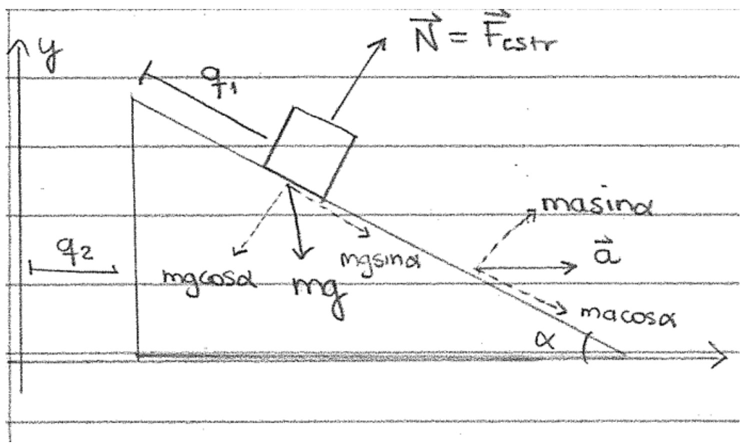

Figure 1: Problem 1

SOLUTION - We start this problem decomposing the forces that are acting on the block into parallel and perpendicular to the wedge components. By Newton's second law, the sum of the forces should be equal to the resulting one, meaning:

$$\sum \mathbf{F}_\perp = m\mathbf{a}_\perp \tag{1}$$

$$F_{cstr} - mg\cos\alpha = ma\sin\alpha \tag{2}$$

Note that our constraint force in this problem is equivalent to the normal force, given by equation (2). In a similar way, to find the equation of motion for the block, we write the parallel component of the forces, taking into account that the wedge and the block are moving together - there's no relative acceleration

$$\sum \mathbf{F}_\parallel = m\mathbf{a}_\parallel = 0 \tag{3}$$

$$mg\sin\alpha = ma\cos\alpha \tag{4}$$

$$\ddot{q}_1 = g\sin\alpha - a\cos\alpha \tag{5}$$

To check this answer: if $a\cos\alpha > g\sin\alpha$, meaning the plane has a higher parallel acceleration than the block, $\ddot{q}_1 < 0$ and the block will slide up the plane.

(b) Set up the equations of motion using Lagrangian methods, with generalized coordinate q_1. Again, check that the equation of motion for $\ddot{q}_1$ is the same as in part a).

SOLUTION - Using the Lagrangian method, we rewrite the Cartesian coordinates x and y in terms of the generalized ones for the block only:

$$x = q_2 + q_1 \cos\alpha \rightarrow \dot{x} = at + \dot{q}_1 \cos\alpha \tag{6}$$
$$y = -q_1 \sin\alpha \rightarrow \dot{y} = -\dot{q}_1 \sin\alpha \tag{7}$$

With this new definitions we can write the kinectic and potential energies:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}(a^2t^2 + 2at\dot{q}_1 \cos\alpha + \dot{q}_1{}^2) \tag{8}$$
$$U = mgy = -mgq_1 \sin\alpha \tag{9}$$

And from the Lagrangian $L = T - U$ we get the same equation of motion for q_1, as expected:

$$ma\cos\alpha + m\ddot{q}_1 = mg\sin\alpha \tag{10}$$
$$\ddot{q}_1 = g\sin\alpha - a\cos\alpha \quad \square \tag{11}$$

2. A particle of mass m moves freely over the surface of the sphere with Lagragian

$$\mathcal{L} = \frac{1}{2}m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2). \tag{12}$$

Show that the Lagrangian and the quantity $p_\phi = mR^2\sin^2(\theta)\dot{\phi}$ are constants of the motion - they are conserved - and give a physical interpretation.

SOLUTION - To prove that the generalized momentum is conserved we look at the Euler-Lagrange equation for ϕ:

$$\frac{\partial L}{\partial \phi} = 0 \rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} = 0 \tag{13}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}(mR^2 sin^2\theta\dot{\phi}) = 0 \rightarrow mR^2\sin^2\theta\dot{\phi} = const. \tag{14}$$

And for the Lagrangian we calculate its total time derivative,

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \dot{\theta}\frac{\partial L}{\partial \theta} + \ddot{\theta}\frac{\partial L}{\partial \dot{\theta}} + \dot{\phi}\frac{\partial L}{\partial \phi} + \ddot{\phi}\frac{\partial L}{\partial \dot{\phi}}. \tag{15}$$

From the right hand side we have:

$$\frac{\partial L}{\partial \theta} = mR^2\sin\theta\cos\theta\dot{\phi}^2 \tag{16}$$
$$\frac{\partial L}{\partial \dot{\theta}} - mR^2\dot{\theta} \tag{17}$$
$$\frac{\partial L}{\partial \dot{\phi}} = mR^2\sin^2\theta\dot{\phi} \tag{18}$$

And from the equation of motion for θ

$$mR^2 \sin\theta\cos\theta\dot{\phi} = mR^2\ddot{\theta} \tag{19}$$

Back to $\mathrm{d}L/\mathrm{d}t$:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = (mR^2\sin\theta\cos\theta\dot{\phi}^2 + mR^2\ddot{\theta})\dot{\theta} + mR^2\sin^2\theta\dot{\phi}\ddot{\phi} \tag{20}$$

$$= mR^2\dot{\phi}(2\sin\theta\cos\theta\dot{\phi}\dot{\theta} + \sin^2\theta\ddot{\phi}) \tag{21}$$

$$= mR^2\dot{\phi}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\phi}} = 0 \tag{22}$$

Given that the Lagrangian does not depend on the ϕ coordinate, we have that the generalized momentum associated with this coordinate p_ϕ is conserved, and here it represents the z component of the angular momentum of the particle. Also, because the Lagrangian does not have an explicit dependence on time, we have that this system conserves its total energy.

3. Determine the degrees of freedom, the kinetic energy, the generalized forces and the equations of motion (you don't need to solve them!) for the following systems in a constant gravitational field:

(a) The double Atwood machine.

SOLUTION - At first, it's possible to identify 4 different coordinates to describe this system. However, a second look reveals that there are 2 constraints that reduce the number of generalized coordinates in half:

$$x_1 + x_4 = l_1 // x_2 + x_3 = l_2 \tag{23}$$

where l_1 and l_2 are the lenghts of the strings on each machine. Choosing x_1 and x_2 as the two degrees of freedom to describe the motion, we can write

$$\dot{x_1} = -\dot{x_4}\dot{x_2} = -\dot{x_3} \tag{24}$$

and the kinectic and potential energy are given by:

$$T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 \tag{25}$$

$$= \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2(\dot{x_2} - \dot{x_1})^2 + \frac{1}{2}m_3(-\dot{x_2} - \dot{x_1})^2 \tag{26}$$

$$= \frac{1}{2}\dot{x_1}^2(m_1 + m_2 + m_3) + \frac{1}{2}\dot{x_2}^2)(m_2 + m_3) + \dot{x_1}\dot{x_2}(m_3 - m_2) \tag{27}$$

$$U = -m_1gx_1 - m_2g(x_2 + x_4) - m_3g(x_3 + x_4) \tag{28}$$

$$= m_1gx_1 - m_2g(x_2 - l_1 - x_1) - m_3g(l_2 - x_2 + l_1 - x_1) \tag{29}$$

$$= -gx_1(m_1 - m_2 - m_3) - gx_2(m_2m_3) + constants \tag{30}$$

Thus its Lagragrangian is

$$L = \frac{1}{2}\dot{x_1}^2(m_1+m_2+m_3)+\frac{1}{2}\dot{x_2}^2(m_2+m_3)+\dot{x_1}\dot{x_2}(m_3-m_2)+gx_1(m_1-m_2-m_3)+gx_2(m_2-m_3) \tag{31}$$

Now, for each coordinate we calculate its Euler-Lagrange equation, starting with x_1:

$$\frac{\partial L}{\partial \dot{x_1}} = \dot{x_1}(m_1 + m_2 + m_3) + \dot{x_2}(m_3 - m_2) \tag{32}$$

$$\frac{\partial L}{\partial x_1} = g(m_1 - m_2 - m_3) \tag{33}$$

$$(m_1 + m_2 + m_3)\ddot{x_1} + (m_3 - m_2)\ddot{x_2} = g(m_1 - m_2 - m_3) \tag{34}$$

And for x_2:

$$\frac{\partial L}{\partial \dot{x_2}} = \dot{x_2}(m_2 + m_3) + \dot{x_1}(m_3 - m_2) \tag{35}$$

$$\frac{\partial L}{\partial x_2} = g(m_2 - m_3) \tag{36}$$

$$(m_2 + m_3)\ddot{x_2} = g(m_2 - m_3) - (m_3 - m_2)\ddot{x_1} \quad \square \tag{37}$$

(b) A mass hanging from a spring with constant k

SOLUTION - Considering only one degree of freedom, the coordinate y, we have:

$$L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2 + mgy \tag{38}$$

and its equation of motion is given by

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y} \tag{39}$$

$$\frac{\partial L}{\partial y} = -ky + mg \tag{40}$$

$$\ddot{y} + \frac{k}{m}y = mg \quad \square \tag{41}$$

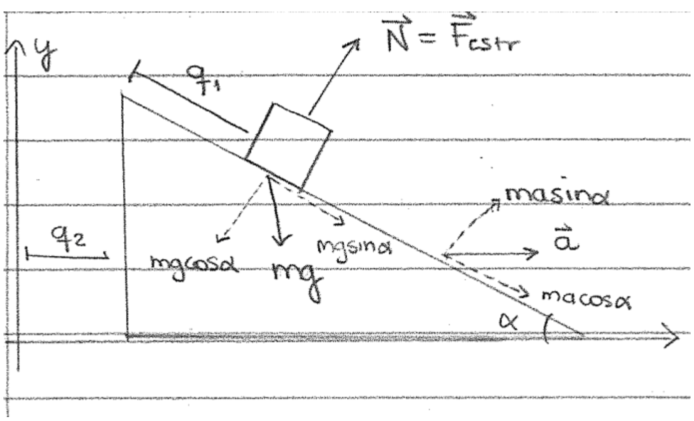

Figure 2: Problem 3a)

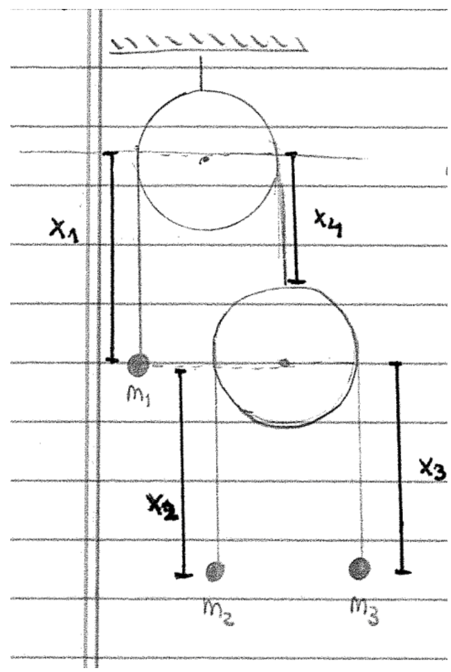

Figure 3: Problem 3b)