Homework # 5

Physics 411 Due: 02.23.2018 Mechanics I

1. In many problems in the calculus of varioations, one needs to determine the lenght ds of a short segment of a curve on a surface (see Eq. (6.1) in the book). Determine ds for the following situations:

(a) Curve given by $y = y(x)$ in a plane

SOLUTION - We always start from Cartesian coordinates, for all parts. For a) and b) it's straight forward:

$$\mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = y'(x)\mathrm{d}x \tag{1}$$

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + [y'(x)]^2}\mathrm{d}x \tag{2}$$

(b) As in a) but $x = x(y)$.

SOLUTION - Again, just calculate ds:

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}y}\mathrm{d}y = x'(y)\mathrm{d}y \tag{3}$$

$$\mathrm{d}s = \sqrt{[x'(y)]^2 + 1}\mathrm{d}y \tag{4}$$

(c) As in a) but $r = r(\phi)$.

SOLUTION - We first make the transformation from Cartesian coordinates to polar coordinates:

$$x = r\cos\phi \rightarrow \mathrm{d}x = -r\sin\phi\mathrm{d}\phi + \cos\phi\mathrm{d}r \tag{5}$$

$$y = r\sin\phi \rightarrow \mathrm{d}y = r\cos\phi\mathrm{d}\phi + \sin\phi\mathrm{d}r \tag{6}$$

Plugging into ds:

$$\mathrm{d}s^2 = r^2\mathrm{d}\phi^2(\sin^2\phi + \cos^2\phi) + \mathrm{d}r^2(\cos^2\phi + \sin^2\phi) \tag{7}$$

$$\mathrm{d}s = \sqrt{r^2 + r'^2}\mathrm{d}\phi \tag{8}$$

(d) As in a) but $\phi = \phi(r)$.

SOLUTION - Using the result from equation 8, we only have to rewrite ϕ:

$$\phi(r) : \mathrm{d}r^2\left[r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) + 1\right] \tag{9}$$

$$\mathrm{d}s = \sqrt{r^2\phi'^2 + 1}\mathrm{d}r \tag{10}$$

(e) Curve given by $\phi = \phi(z)$ on a cylinder of radius R.

SOLUTION - In this new coordinate system we add z as an extra variable, and set $r = R$, meaning $\mathrm{d}r = 0$:

$$x = r\cos\phi \rightarrow \mathrm{d}x = -R\sin\phi\mathrm{d}\phi \quad (11)$$
$$y = r\sin\phi \rightarrow \mathrm{d}y = R\cos\phi\mathrm{d}\phi \quad (12)$$
$$z = z \rightarrow \mathrm{d}z = \mathrm{d}z \quad (13)$$

Thus, $\mathrm{d}s$ is

$$\mathrm{d}s = \sqrt{R^2\mathrm{d}\phi^2(\sin^2\phi + \cos^2\phi) + \mathrm{d}z^2} \quad (14)$$
$$= \sqrt{R^2\mathrm{d}\phi^2 + \mathrm{d}z^2} \quad (15)$$

Using that $\phi = \phi(z)$

$$\mathrm{d}s = \sqrt{R^2\phi'^2 + 1}\mathrm{d}z \quad (16)$$

(f) As in e) but $z = z(\phi)$.

SOLUTION - Starting from equation 14:

$$\mathrm{d}s = \sqrt{\mathrm{d}\phi^2[R^2 + (\mathrm{d}z/\mathrm{d}\phi)^2]} \quad (17)$$
$$= \sqrt{R^2 + z'^2}\mathrm{d}\phi \quad (18)$$

(g) Curve given by $\theta = \theta(\phi)$ on a sphere of radius R.

SOLUTION - Here we transform x, y and z into r, ϕ and θ, taking into account that $r = R$ and $\mathrm{d}r = 0$:

$$x = r\cos\phi\sin\theta \rightarrow \mathrm{d}x = -R\sin\phi\sin\theta\mathrm{d}\phi + R\cos\phi\cos\theta\mathrm{d}\theta \quad (19)$$
$$y = r\sin\phi\sin\theta \rightarrow \mathrm{d}y = R\cos\phi\sin\theta\mathrm{d}\phi + R\sin\phi\cos\theta\mathrm{d}\theta \quad (20)$$
$$z = r\cos\theta \rightarrow \mathrm{d}z = -R\sin\theta\mathrm{d}\theta \quad (21)$$

Resulting in a $\mathrm{d}s$ equal to

$$\mathrm{d}s^2 = R^2\sin^\theta\mathrm{d}\phi^2(\sin^2\phi + \cos^2\phi) + R^2\cos^2\theta\mathrm{d}\theta^2(\cos^2\phi + \sin^2\phi) + R^2\sin^2\theta\mathrm{d}\theta \quad (22)$$
$$= R^2\sin^2\theta\mathrm{d}\phi^2 + R^2\cos^2\theta\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\theta^2 \quad (23)$$
$$= R^2\sin^2\theta\mathrm{d}\phi^2 + R^2\mathrm{d}\theta^2 \quad (24)$$
$$\rightarrow \mathrm{d}s = \sqrt{R^2\mathrm{d}\phi^2[\sin^2\theta + (\mathrm{d}\theta/\mathrm{d}\phi)^2]} \quad (25)$$
$$\mathrm{d}s = \sqrt{\sin^2\theta + \theta'^2}R\mathrm{d}\phi \quad (26)$$

(h) As in g) but $\phi = phi(\theta)$.

SOLUTION - From equation 24:

$$\mathrm{d}s = \sqrt{R^2 \sin^2 \theta \mathrm{d}\phi^2 + R^2 \mathrm{d}\theta^2} \tag{27}$$

$$= \sqrt{\mathrm{d}\theta^2 [R^2 \sin^2 \theta (\mathrm{d}\phi/\mathrm{d}\theta)^2 + R^2]} \tag{28}$$

$$= \sqrt{\sin^2 \theta \phi'^2 + 1} R \mathrm{d}\theta \tag{29}$$

2. The shortest path between two points on a *curved surface*, such as the surface of a sphere is called a **geodesic**. To find a geodesic, one finds the curve that makes the path lenght stationary. In a plane, we saw that $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}dx$.

(a) Use spherical polar coordinates (r, θ, ϕ) to show that the lenght of the path joining two points on a sphere of radius R is

$$L = R \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^\theta \phi'^2} d\theta. \tag{30}$$

SOLUTION - From part h) in the previous problem we have ds as a function of θ, thus the lenght between two points can be written as

$$L = \int_{p_1}^{p_2} = R \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^\theta \phi'^2} d\theta \tag{31}$$

(b) Prove that the geodecis between two given points on a sphere is a great circle. Note that the integrand above is independent of ϕ, so the Euler-Lagrange equations reduce to $\partial f / \partial \phi' = C$, where C is a constant. You can always choose your z axis to pass through the point 1, which will allow you to show that $C = 0$ in this case. Describe the corresponding geodesics (Note that with this choice of axez, the point 1 is at the North pole.)

SOLUTION - we have an integral of the form

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x] dx \tag{32}$$

and in our case S is the lenght we want to minimize with an integrand equal to ds = d$s(\phi'(\theta), \theta)$. The condition to ensure that L is minimized is

$$\underbrace{\frac{\partial f}{\partial \phi}}_{=0} - \frac{\mathrm{d}}{\mathrm{d}\theta} \frac{\partial f}{\partial \phi'} = 0 \tag{33}$$

which gives

$$\frac{\sin^2 \theta \phi'}{(1 + \sin^2 \theta \phi'^2)} = C \tag{34}$$

This equation is valid for any θ: choosing a coordinate system that the z-axis pass through point 1 means that for θ_1 this expression is equal to zero

$$\frac{\sin^2(0)\phi'2}{(1+sin^2(0)\phi'^2)} = 0 \tag{35}$$

Thereforre $C = 0$ for all values of θ. The only way for equation 34 be true for any ϕ and θ is that we have $\phi' = 0$ which means ϕ is a constant. On a sphere, keeping ϕ constant means that the only degree of freedom is θ, so the shortest path between two points in it is a great circle.

3. A pendulum bob of mass m is suspended by a string of lenght l from a point of support. The point of support moves back and forth along the horizontal x-axis according to the equation $x = a\cos(\omega t)$, with a a constant. Assujme that the pendulum's position remains in the x-z plane (z is vertical), and describe the pendulum's position by the angle θ that the string makes with the vertical.

(a) Set up the Lagragian, and from it find the equation of motion, using the single generalized coordinate θ.

SOLUTION - For any problems with generalized coordinates, the reference frame will determine how we transform for Cartesian to the new system. Regardless of choice, the equation of motion should be the same since we're describing the same problem. Defining x and z in terms of θ:

$$x(t) = a\cos(\omega t) + l\sin(\theta) \to \dot{x}(t) = -a\sin(\omega t) + l\dot{\theta}\cos(\omega t) \tag{36}$$

$$z(t) = -l\cos(\theta) \to l\dot{\theta}\sin(\theta) \tag{37}$$

(Note that θ is a fuction of time!) With these definition we can write the potential and kinectic energy for this system:

$$U = mgz = -mgl\cos(\theta) \tag{38}$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{z}^2) = \frac{1}{2}m(a^2\omega^2\sin^2(\omega t) + l^2\dot{\theta}^2 - 2a\omega l\dot{\theta}\sin(\omega t)\cos(\theta)) \tag{39}$$

and its Lagrangian

$$\mathscr{L} = \frac{1}{2}m(a^2\omega^2\sin^2(\omega t) + l^2\dot{\theta}^2 - 2a\omega l\dot{\theta}\sin(\omega t)\cos(\theta)) + mgl\cos(\omega t) \tag{40}$$

To find the equation of motion for θ we need to calculate the following derivatives:

$$\frac{\partial\mathscr{L}}{\partial\theta} = ml\sin(\theta)(a\omega\dot{\theta}\sin(\omega t) - g) \tag{41}$$

$$\frac{\partial\mathscr{L}}{\partial\dot{\theta}} = ml^2\dot{\theta} - ma\omega l\sin(\omega t)\cos(\theta) \tag{42}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathscr{L}}{\partial\theta} = ml^2\ddot{\theta} - ma\omega^2 l\cos(\omega t)\cos(\theta) + ma\omega l\dot{\theta}\sin(\omega t)\sin(\theta) \tag{43}$$

which gives the Euler-Lagrange equation

$$ml^2\ddot{\theta} + mgl\sin(\theta) - ma\omega^2 l\cos(\omega t)\cos(\theta) = 0 \quad \square \tag{44}$$

(b) Show that, for small values of θ, the equation reduces to that of a driven harmonic oscillator and find the corresponding steady state motion.

SOLUTION - From the Taylor's series approximation, we have that for small angles, equation 44 can be simplified to

$$ml^2\ddot{\theta} + mgl\theta - ma\omega^2 l\cos(\omega t) = 0 \tag{45}$$

$$\ddot{\theta} + \omega_0^2\theta = \frac{a\omega^2}{l}\cos(\omega t) \tag{46}$$

where we recognized the natural frequency of this system, $\omega_0^2 = g/l$. Now that we have the equation for a driven harmonic oscillator, it's possible to find its steady state solution starting with guess such as $\theta(t) = \theta_0\cos(\omega t)$. Plugging this expression into the equation of motion we obtain

$$-\omega^2\theta_O\cos(\omega t) + \omega_0^2\theta_0\cos(\omega t) = \frac{a}{l}\omega^2\cos(\omega t) \tag{47}$$

$$(\omega_0^2 - \omega^2)\theta_0 = \frac{a}{l}\omega^2 \;\; \rightarrow \theta_0 = \frac{a}{l}\frac{\omega^2}{\omega_0^2 - \omega^2} \tag{48}$$

and finally we have

$$\theta(t) = \frac{a}{l}\frac{\omega^2}{\omega_0^2 - \omega^2}\cos(\omega t) \quad \square \tag{49}$$

(c) Comment on the use of such a device as a seismograph to sense horizontal oscillations of the Earth surface. Given the choice, would it be better to have the pendulums natural frequency much greater or much less than the typical vibrational frequencies of the Earth?

SOLUTION - If $\omega_0 \gg \omega$,

$$\theta(t) \approx \frac{a}{l}\frac{\omega^2}{\omega_0^2}\cos(\omega t) \tag{50}$$

which means the amplitude would be very small and also dependent on ω. Not what we are looking for in a seismograph. On the other hand, if $\omega \gg \omega_0$,

$$\theta(t) \approx \frac{a}{l}\cos(\omega t) \tag{51}$$

which gives us an amplitude that is independent of ω(Earth's frequency) and has a clear relation with it through a well behaved function, making it accurate for a seismograph set up.