1. A meteorite falls to earth from very far away with negligible initial speed. Neglecting air friction, how fast is it going when it hits the earth's surface? Express your answer in terms of the mass of the earth M_E, the radius of the earth R_E, and the gravitational constant G. You may take the initial distance from the earth to be infinite.

SOLUTION - We can solve this problem using conservation of energy. The potential energy for the meteorite can be calculated from Earth's gravitational force as follows

$$U(r) = -\int_{\infty}^{r} \mathbf{F}(r') \cdot \mathrm{d}\mathbf{r}' \tag{1}$$

$$= -\int_{\infty}^{r} \frac{GmM_E}{r^2}\hat{\mathbf{r}} \cdot \mathbf{r}' \tag{2}$$

$$= -\frac{GmM_E}{r} \tag{3}$$

At the initial point, the total energy is equal to zero - its speed it's negligible which means no kinectic energy, and it's at a infinite distance from Earth which means it's not affected by its gravitational pull. The final energy is given by

$$E_{final} = T_{final} + U(R_E) = \frac{1}{2}mv_{final}^2 - \frac{GmM_E}{R_E} \tag{4}$$

and it's possible to find its final velocity by equating initial and final energy:

$$\frac{1}{2}mv_{final}^2 - \frac{GmM_E}{R_E} = 0 \tag{5}$$

$$v_{final} = \sqrt{\frac{2GM_E}{R_E}} \quad \square \tag{6}$$

2. Suppose a particle of mass m moves along the x-axis governed by a force $F(x) = -ax+bx^3$, where a and b are positive constants.

(a) Find the potential $U(x)$ and draw a rough graph of it.

SOLUTION - The potential energy is given by

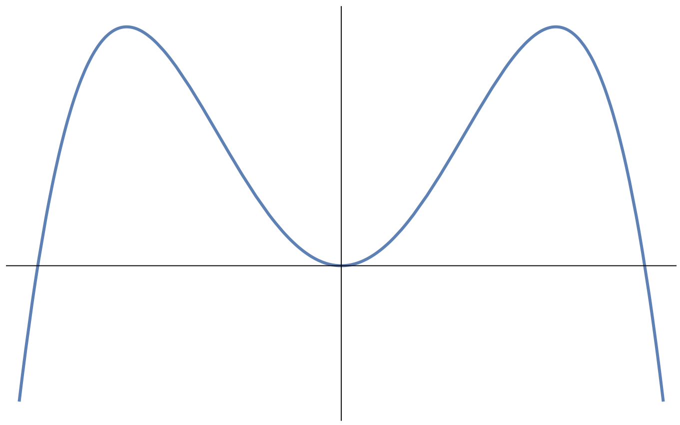
$$U(x) = -\int F(x)dx = \frac{1}{2}ax^2 - \frac{1}{4}bx^4. \tag{7}$$

To make a a rough plot of this, it's useful to know its extreme points

$$\frac{du}{dx} = ax - bx^3 = 0 \tag{8}$$

$$\rightarrow x(a - bx^2) = 0 \tag{9}$$

This gives us three inflection points, two maxima at $x = \pm\sqrt{a/b}$ and a minimum at $x = 0$ and wil have a general form like this:



(b) What is the minimum speed the particle must be given at $x = 0$ so that it will escape to $x = +\infty$?

SOLUTION - To escape to $x = \infty$ the particle must overcome the potential barrier which has a maximum at $x = \sqrt{a/b}$. We start from the origin and would like for it have enough energy to get to that point, so we can assume that at $x = \sqrt{a/b}$ its velocity has to be greater or equal to zero. Using conservation of energy we find

$$E_{inital} = T_{initial} + U(x=0) = E_{final} = T(v=0) + U(x = \sqrt{a/b}) \tag{10}$$

$$\frac{1}{2}mv^2 = \frac{1}{2}a\left(\frac{a}{b}\right) - \frac{1}{4}b\left(\frac{a}{b}\right)^2 \tag{11}$$

$$\frac{1}{2}mv^2 = \frac{1}{2}\frac{a^2}{b} \tag{12}$$

$$v = \sqrt{\frac{a^2}{2mb}} \quad \square \tag{13}$$

3. A very useful application of the gradient is that it gives us the change in U (or any scalar function) resulting from a small displacement $d\mathbf{r}$:

$$\mathrm{d}U = \boldsymbol{\nabla} U \cdot \mathrm{d}\mathbf{r} \tag{14}$$

(a) Show that the direction of $\boldsymbol{\nabla} U$ at any point $\mathbf{r}$ is the direction in which U increases fastest as we move away from $\mathbf{r}$.(Choose a small displacement $d\mathbf{r} = \epsilon u$, where $\mathbf{u}$ is a unit vector and ϵ is fixed and small, and find the direction of $\mathbf{u}$ for which dU is largest).

SOLUTION - Rewriting equation 14 using $d\mathbf{r} = \epsilon\hat{\mathbf{u}}$, we have

$$\mathrm{d}U = \nabla U \cdot \epsilon\hat{\mathbf{u}} = \epsilon \nabla U \cdot \hat{\mathbf{u}} \tag{15}$$

$$= \epsilon|\nabla U| \underbrace{|u|}_{=1} \cos\theta \tag{16}$$

$$= \epsilon|\nabla U| \cos\theta \tag{17}$$

Therefore, dU is at its maximum when $\cos\theta = 1 \rightarrow \theta = 0$, which happens when ∇U and $d\mathbf{r}$ are parallel. This is the direction that U increases fastest as we move away from $\mathbf{r}$.

(b) Which of the following forces is conservative? (i) $\mathbf{F} = k(x, 2y, 3z)$ where k is a constant. (ii) $\mathbf{F} = k(y, x, 0)$. (iii) $\mathbf{F} = k(-y, x, 0)$. For those which are conservative, find the corresponding potential energy U, and verify by direct differentiation that $\mathbf{F} = \boldsymbol{\nabla} U$.

SOLUTION - We can determine that a force is conservative by calculating its curl.

(i) $\mathbf{F} = k(x, 2y, 3z)$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ kx & 2ky & 3kz \end{vmatrix} \tag{18}$$

$$= \frac{\partial}{\partial y}(3kz)\hat{\mathbf{i}} + \frac{\partial}{\partial z}(2ky)\hat{\mathbf{k}} - \frac{\partial}{\partial y}(kx)\hat{\mathbf{k}} - \frac{\partial}{\partial x}(3kz)\hat{\mathbf{j}} - \frac{\partial}{\partial z}(2ky)\hat{\mathbf{i}} \tag{19}$$

This force is conservative because $\nabla \times \mathbf{F} = 0$. By direct integration we can find the potential energy:

$$U = -\int \mathbf{F} \cdot \mathbf{r} = -\int F_x dx' - \int F_y dy' - \int F_z dz' \tag{20}$$

$$= -k \int x' dx' - 2k \int y' dy' - 3k \int 3z' dz' \tag{21}$$

$$= -k \left(\frac{1}{2}x^2 + y^2 + \frac{3}{2}z^2 \right) \tag{22}$$

and check that $\mathbf{F} = -\nabla U$

$$\mathbf{F} = k \left[\frac{\partial}{\partial x} \left(\frac{1}{2}x^2 \right) \hat{\mathbf{i}} + \frac{\partial}{\partial y} y^2 \hat{\mathbf{j}} + \frac{\partial}{\partial z} \left(\frac{3}{2}z^2 \right) \hat{\mathbf{k}} \right] \tag{23}$$

$$= k(x, 2y, 3z) \quad \square \tag{24}$$

(ii) $\mathbf{F} = k(y, x, 0)$

$$\nabla \times \mathbf{F} = \frac{\partial}{\partial z}(ky)\hat{\mathbf{j}} + \frac{\partial}{\partial x}(kx)\hat{\mathbf{k}} + \frac{\partial}{\partial y}(ky)\hat{\mathbf{k}} + \frac{\partial}{\partial z}(kx)\hat{\mathbf{i}} = 0 \tag{25}$$

$$U = -k \int y' dx' + k \int x' dy' = -k \int \mathrm{d}(x'y') = -kxy \tag{26}$$

$$\mathbf{F} = -\nabla U = k \left[\frac{\partial}{\partial x}(yx)\hat{\mathbf{i}} + \frac{\partial}{\partial y}(yx)\hat{\mathbf{j}} + \frac{\partial}{\partial z}(yx)\hat{\mathbf{k}} \right] \tag{27}$$

$$= k(y, x, 0) \quad \square \tag{28}$$

(iii) $\mathbf{F} = k(-y, x, 0)$

$$\nabla \times \mathbf{F} = \frac{\partial}{\partial z}(ky)\hat{\mathbf{j}} + \frac{\partial}{\partial x}(kx)\hat{\mathbf{k}} - \frac{\partial}{\partial y}(ky)\hat{\mathbf{k}} + \frac{\partial}{\partial z}(kx)\hat{\mathbf{i}} = 2k\hat{\mathbf{k}} \tag{29}$$

This force is not conservative and $\mathbf{F} = -\nabla U$ does not hold true in this case.

4. Consider a mass m confined to the x-axis with a potential energy of $U = kx^4$ with $k > 0$. At time $t = 0$, when it is sitting at the origin, it is given a sudden kick to the right.

(a) Find the time for the mass to reach its maximum displacement $x_{max} = A$. Your answer will be given as an integral over x in terms of m, A and k.

SOLUTION - Using conservation of energy, we have

$$E_{initial} = E_{final} \tag{30}$$

$$\frac{1}{2}mv^2 + kx^4 = kA^4 \tag{31}$$

given that at x_{max} the energy is purely potential. To find t_{max} we integrate the expression found for $v(t)$ as follows

$$\frac{1}{2}mv^2 = k(A^4 - x^4) \tag{32}$$

$$\frac{dx}{dt} = \left[\frac{2k}{m}(A^4 - x^4)\right] \tag{33}$$

$$\int_0^{t_{max}} dt = \int_0^A \sqrt{\frac{m}{2k}} \frac{dx}{(A^4 - x^4)^{1/2}} \tag{34}$$

$$t_{max} = \sqrt{\frac{m}{2k}} \int_0^A \frac{dx}{(A^4 - x^4)^{1/2}} \quad \square \tag{35}$$

(b) Find the period τ of oscillations of amplitude A, and by making a suitable change of variables, show that it is inversely proportional to A. (Thus, the larger the amplitude the shorter the period!)

SOLUTION - Defining $y = x/A$, we can rewrite t_{max} as

$$t_{max} = \sqrt{\frac{m}{2k}} \int_0^A \frac{dx}{A^2(1 - x^4/A^4)^{1/2}} \tag{36}$$

$$= \sqrt{\frac{m}{2k}} \int_0^A \frac{Ady}{A^2(1 - y^4)^{1/2}} \tag{37}$$

This integral can be solved numerically and since it's a definite integral it's just a number. Because t_{max} gives the time it takes to go from the origin to x_{max} which corresponds to 1/4 of the complete oscillation, the period of oscillation is just $4t_{max}$. (Think about how many "t_{max}" does it take for the particle to go from $-x_{max}$ to x_{max} and back.)

(c) The integral cannot be evaluated in terms of elementary functions. This is often the case, but for small oscillations about the minimum of any potential energy $U(x)$, we may approximate U by the first three terms of its Taylor series in powers of x (or if the minimum was located at $x = a$, in powers of $x - a$). Write down the equation of conservation of energy. The small oscillations about the minimum are thus approximately simple harmonic. If

$$U = \frac{x(x-3)^2}{3}, \tag{38}$$

find the period of small oscillations about the minimum at $x = 3$. (Note that there is another equilibrium point at $x = 1$, but it is a maximum.)

SOLUTION - First we expand $U(x)$ in a Tylor series around $x = 3$. A Taylor series looks like this for a given function $f(x)$:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} + \frac{f''(x_0)}{2!} + \frac{f'''(x_0)}{3!} + \cdots \tag{39}$$

Thus for this potential energy we find

$$U(x=3) = 0 \tag{40}$$

$$U'(x=3) = \frac{(x-3)^2}{3} + \frac{2x(x-3)}{3}\bigg|_{x=3} = 0 \tag{41}$$

$$U''(x=3) = \frac{2(x-3)}{3} + \frac{2x}{3}\bigg|_{x=3} = 2 \tag{42}$$

Therefore its Taylor series is $U(x) \approx \frac{2}{2!}(x - 3^2) = (x-3)^2$. This approximation for small oscillations has the same form of a harmonic oscillator $U(x) = \frac{1}{2}kx^2$ with $k = 2$, so the period is given by $\tau = 2\pi\sqrt{m/k}$

$$\tau = 2\pi\sqrt{\frac{m}{2}} = \pi\sqrt{2m} \quad \square \tag{43}$$