1. A projectile is fired from the origin of a coordinate system, in the x-y plane (x is the horizontal displacement; y, the vertical) with initial velocity $v_0 = (v_{0x}, v_{0y}, 0)$. Consider only the force of gravity on this projectile.

(a) Find the components of the velocity (v_x and v_y) and the components of the displacement (x and y) as functions of time.

SOLUTION - We start writing Newton's second law for x and y components:

$$\mathbf{F_x} = 0, \tag{1}$$
$$\mathbf{F_y} = -mg \tag{2}$$

which will give us v_x and v_y

$$\dot{v_x}(t) = 0 \rightarrow v_x(t) = v_{0x} \tag{3}$$
$$\dot{v_y}(t) = -mg \rightarrow v_y(t) = v_{0y} - gt \tag{4}$$

and also r_x and r_y

$$\dot{r_x}(t) = v_{0x} \rightarrow r_x(t) = v_{0x}t \tag{5}$$
$$\dot{r_y}(t) = v_{0y} - gt \rightarrow r_y(t) = v_{0y}t - \frac{1}{2}gt^2 \quad \square \tag{6}$$

(b) Find, as a function of time, the torque $\mathbf{N}$ about the origin which is exerted by gravity on the projectile. Give magnitude and direction.

SOLUTION - The torque exerted by gravity is given by

$$\mathbf{N} = \mathbf{r} \times \mathbf{F} \tag{7}$$
$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_x t & v_y t - \frac{1}{2}gt^2 & 0 \\ 0 & -mg & 0 \end{vmatrix} \tag{8}$$
$$= -mgv_x t\hat{\mathbf{k}} \quad \square \tag{9}$$

(c) Find, as a function of time, the angular momentum $\mathbf{L}$ of the particle about the origin. Give magnitude and direction.

SOLUTION - Similar procedure for $\mathbf{L}$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v}) \tag{10}$$
$$= m\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_x t & v_y t - \frac{1}{2}gt^2 & 0 \\ v_x & v_y t - gt & 0 \end{vmatrix} \tag{11}$$
$$= mv_x t(v_y - gt)\hat{\mathbf{k}} - mv_x(v_y t - \frac{1}{2}gt^2)\hat{\mathbf{k}} \tag{12}$$
$$= -\frac{1}{2}mv_x gt^2\hat{\mathbf{k}} \quad \square \tag{13}$$

(d) Check that $\mathbf{N} = \frac{d\mathbf{L}}{dt}$.

SOLUTION - By direct calculation we find

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = -mgv_x t\hat{\mathbf{k}} = \mathbf{N} \quad \square \tag{14}$$

2. A particle is projected vertically upward in a constant gravitational field with an initial speed v_0. There is a drag force proportional to the square of the instantaneous speed.

(a) Write Newton's second law for the upward motion and integrate it to find the velocity and the height, $v(t)$, $y(t)$. Show that the particle reaches an altitude

$$y_{max} = \frac{v_{ter}^2}{g} \log\left(\sqrt{1 + v_0^2/v_{ter}^2}\right) \tag{15}$$

where $v_t er = \sqrt{mg/c}$ is the terminal velocity for the *downward* motion.

SOLUTION - For this problem, gravity is defined as a positive constant, along the negative y direction. From Newton's second law, we have the net force on the rocket

$$|\mathbf{F}| = m\frac{dv}{dt} = -mg - cv^2 \tag{16}$$

Solving for $v(t)$ and defining terminal velocity as $v_{ter} = \sqrt{mg/c}$:

$$\int_{v_0}^{v} \frac{\mathrm{d}v'}{(1 + v'^2/v_{ter}^2)} = \int_0^t gt'dt \tag{17}$$

On the right hand side of equation 17, we can use the following integral to find a solution

$$\int \frac{dx}{1 + ax^2} = \frac{1}{\sqrt{a}} \arctan\left(\sqrt{a}x\right) + \text{constant} \tag{18}$$

Thus we can write

$$\arctan\left(\frac{v}{v_{ter}}\right) - \arctan\left(\frac{v_0}{v_{ter}}\right) = -\frac{gt}{v_{ter}} \tag{19}$$

$$v(t) = v_{ter}\tan\left(\arctan\left(\frac{v_0}{v_{ter}}\right) - \frac{gt}{v_{ter}}\right) \quad \square \tag{20}$$

We can integrate the expression above to find $y(t)$

$$y(t) - 0 = v_{ter} \int_0^t \tan\left(\arctan\left(\frac{v_0}{v_{ter}}\right) - \frac{gt}{v_{ter}}\right) dt' \tag{21}$$

$$= \frac{v_{ter}^2}{g} \left\{ \log\left[\cos\left(\arctan\left(\frac{v_0}{v_{ter}}\right) - \frac{gt}{v_{ter}}\right)\right]\Bigg|_0^t \right\} \tag{22}$$

$$= \frac{v_{ter}^2}{g} \left\{ \log\left[\frac{\cos(\arctan(v_0/v_{ter}) - gt/v_{ter})}{\cos(\arctan(v_0/v_{ter}))}\right]\right\} \tag{23}$$

$$= \frac{v_{ter}^2}{g} \left\{ \log\left[\sqrt{1 + v_0^2/v_{ter}^2} \cos\left(\arctan\left(\frac{v_0}{v_{ter}}\right) - \frac{gt}{v_{ter}}\right)\right]\right\} \quad \square \tag{24}$$

using that $\cos(\arctan(x)) = 1/\sqrt{1+x^2}$ on the second to last equation. To find y_{max} we use that at this point its velocity will be zero,

$$v(t_{max}) = v_{ter} \tan \underbrace{\left(\arctan\left(\frac{v_0}{v_{ter}}\right) - gt\right)}_{=0} = 0 \tag{25}$$

and applying this condition to $y(t)$ we have

$$y_{max} = \frac{v_{ter}^2}{g} \left[\log\left(\sqrt{1 + v_0^2/v_{ter}^2} \cos(0)\right)\right] \tag{26}$$

$$= \frac{v_{ter}^2}{g} \log\left(\sqrt{1 + v_0^2/v_{ter}^2}\right) \quad \square \tag{27}$$

(b) Show that when the particle returns to the initial position, its speed is $v_0 v_t er/\sqrt{v_0^2 + v_t er^2}$.

SOLUTION - For the downwards motion we switch signs on the drag acceleration term, resulting in a slightly different initial equation:

$$|\mathbf{F}| = \frac{dv}{dt} = -g + \frac{v^2}{v_{ter}^2} \tag{28}$$

Solving this equation yields an expression for $v(t)$ and after one more integration $y(t)$, that can be used to find the time t_{final} at which the projectile returns to its inital position. With this is possible to calculate v_{final}. This problem is similar to the one solved in pages 60-61 of the textbook. From 2.57 and 2.58:

$$v(t) = v_{ter} \tanh\left(\frac{gt}{v_{ter}}\right) \tag{29}$$

$$y(t) = \frac{v_{ter}^2}{g} \log\left[\cosh\left(\frac{gt}{v_{ter}}\right)\right] \tag{30}$$

Solving for $t = t_{final}$

$$\frac{v_{ter}^2}{g} \log\left[\sqrt{1 + v_0^2/v_{ter}^2}\right] = \frac{v_{ter}^2}{g} \log\left[\cosh\left(\frac{g t_{final}}{v_{ter}}\right)\right] \tag{31}$$

$$\sqrt{1 + v_0^2/v_{ter}^2} = \cosh\left(\frac{g t_{final}}{v_{ter}}\right) \tag{32}$$

$$t_{final} = \frac{v_{ter}}{g} \text{arccosh}(\sqrt{1 + v_0^2/v_{ter}^2}) \quad \square \tag{33}$$

So its velocity when it returns to the inital position is

$$v(t_{final}) = v_{ter} \tanh\left[\text{arccosh}\left(\sqrt{1 + \frac{v_0^2}{v_{ter}^2}}\right)\right] \tag{34}$$

$$= v_{ter} \frac{1}{\sqrt{1 + v_0^2/v_{ter}^2}} \sqrt{\frac{v_0^2}{v_{ter}^2}} = \frac{v_0 v_{ter}}{\sqrt{v_0^2 + v_{ter}^2}} \tag{35}$$

after using the trigonometric identity $\tanh(\text{arccosh}(x)) = \sqrt{x^2 - 1}/x \quad \square$.

3. Consider a rocket taking off vertically from rest in a gravitational field g. Assume that the rocket ejects fuel at a constant rate, $\dot{m} = -\mu$ (where μ is a positive constant), so that $m = m_0 - \mu t$. The exhaust speed of the fuel with respect to the rocket is also constant, $v_e x$.

(a) Write the equation of motion, and solve for v as a function of t, using separation of variables.

SOLUTION - Starting with the equation of motion, from the change in the momentum

$$\frac{dp}{dt} = -mg = m\frac{dv}{dt} + v_{ex} \underbrace{\frac{dm}{dt}}_{-\mu} \tag{36}$$

Integrating this expression we can find $v(t)$

$$\frac{dv}{dt} = \mu v_{ex} - mg \tag{37}$$

$$\int_0^v \mathrm{d}v' = \int_0^t \underbrace{\left(\frac{\mu v_{ex}}{m} - g\right)}_{-1/b \log(a - bx)} \mathrm{d}t' \tag{38}$$

$$v(t) = v_{ex} \log\left(\frac{m_0}{m_0 - \mu t}\right) - gt \quad \square \tag{39}$$

(b) Describe what would happen if the thrust were smaller than the weight of the rocket.

SOLUTION - Starting from rest at the ground level, having a thrust that is smaller than its weight will cause the rocket to stay in place exausting fuel until its mass decreases enough so the thrust can be greater than the gravitational pull.

(c) Integrate $v(t)$ and show that the rocket's height as a function of t is

$$y(t) = v_{ext}t - \frac{1}{2}gt^2 - \frac{mv_{ext}}{\mu}\log\left(\frac{m_0}{m}\right). \tag{40}$$

SOLUTION - Integrating $v(t)$

$$y(t) - y(0) = \int_t^0 v_{ex}\left[\log\left(\frac{m_0}{m_0 - \mu t'}\right)\right] \mathrm{d}t' \tag{41}$$

$$y(t) = -\frac{1}{2}gt^2 + v_{ex}\left[t\log(m_0) + \int_{m_0}^{m} \log(m'(t))\frac{\mathrm{d}m'}{\mu}\right] \tag{42}$$

$$y(t) = -\frac{1}{2}gt^2 + v_{ex}t\log(m_0) + \frac{v_{ex}}{\mu}(m\log(m) - m - m_0\log(m_0) + m_0). \tag{43}$$

Rewriting the last expression yields

$$y(t) = -\frac{1}{2}gt^2 + v_{ex}t - \frac{v_{ex}m}{\mu}\log\left(\frac{m_0}{m}\right) \quad \square \tag{44}$$

4. A particle of mass m can slide on a smooth horizontal table. The particle is connected to a light inextensible string which passes through a small smooth hole O in the table, so that the lower end of the string hangs vertically below the table. Initially the string is held fixed with the particle moving with speed v_0 on a circle of radius a.

(a) Calculate the tension in the string, and the angular momentum with respect to O.

SOLUTION - The tension on the string has to be equal to the centripetal force, so

$$|\mathbf{T}| = m\frac{v_0^2}{a}, \tag{45}$$

where the constants a and v_0 were given in the problem. For the angular momentum we have

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = (a\hat{\mathbf{i}}) \times (mv_0\hat{\mathbf{j}}) = mav_0\hat{\mathbf{k}} \tag{46}$$

(b) The string is now pulled down from below in such a way that the length between O and the particle is $r(t)$. What is the value of the angular momentum with respect to O as a function of time?

SOLUTION - Given that there's only the tension force acting the particle we know that angular momentum is conserved. It's possible to see that when we calculate the variation o f **L** over time:

$$\frac{d\mathbf{L}}{dt} = \mathbf{N} = \underbrace{\mathbf{r} \times \mathbf{T}}_{\text{parallel}} = 0 \tag{47}$$

meaning that as $r(t)$ changes over time its velocity will also change to compensate that and keep **L** constant. Thus, $\mathbf{L}(t) = mav_0\hat{\mathbf{k}}$.

(c) Use Kepler's Second Law to estimate qualitatively the velocity of the particle as it gets closer to the hole, O. Why are you allowed to use Kepler's Law? Do you think that it would be possible to pull the particle through the hole?

SOLUTION - We can apply Kepler's second law because this problem also deals with central forces and as a consequence this system conserves angular momentum over time. Thinking about it in terms of the area the particle covers, as the string gets shorter the particle velocity will have to increase proportionally to sweep the same area in certain period of time. In the limit of $r \to 0$ its velocity would have to go to infinity requiring an inmensuable amount of energy to do so. Therefore, to pull it through the hole while keeping the angular momentum constant it is not possible.