1. Problem 10.12 from Taylor

(a) SOLUTION - Figure 1 shows a (rough) sketch of a triangular prism centered at the origin of its coordinate system. The z axis is coming out of the page. We calculate the moment of inertia for rotation about this axis using the following equation:

$$\mathcal{I}_{ZZ} = \int (x^2 + y^2)\rho \mathrm{d}V \tag{1}$$

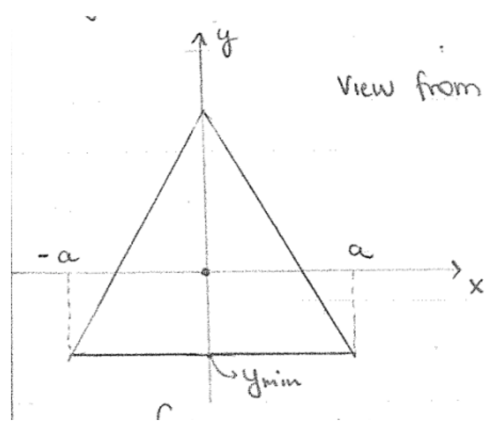

Figure 1: Triangular prism

To determine the limits of integration, it's important to know that in this reference frame the prism is symmetric with respect to the y axis, simplifying the x limits: we can integrate from 0 to a and multiply the result by 2. For the y component, it varies from a y_{min} to the edge of the triangle. This can be described by a linear equation, but first let's detemrine y_{min}. From figure 2, using the know angles for a equilateral triangle we find that $y_{min} = \frac{-\sqrt{3}a}{3}$.

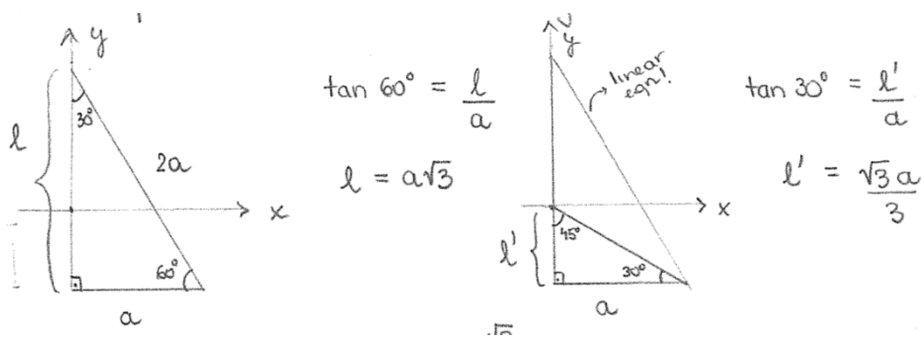

Figure 2: y_{min}

Now, let's determine the linear equation that describes the triangle's hypotenuse:

$$y = mx + b \tag{2}$$

$$y(0) = \frac{2a\sqrt{3}}{3} = b \tag{3}$$

$$m = \tan 60^\circ = \sqrt{3} \tag{4}$$

Finally we set up the integral, using symmetric limits for z:

$$\mathcal{I}_{ZZ} = 2\rho \int_{-h}^{h} \int_{0}^{a} \int_{-\sqrt{3}a/-\sqrt{3}}^{-\sqrt{3}x+2a\sqrt{3}/3} (x^2 + y^2)\mathrm{d}y\mathrm{d}x\mathrm{d}z \tag{5}$$

$$= 2\rho \int_{-h}^{h} \int_{0}^{a} \left[x^2 y + \frac{y^3}{3} \Bigg|_{-\sqrt{3}a/-\sqrt{3}}^{-\sqrt{3}x+2a\sqrt{3}/3} \right] \mathrm{d}x\mathrm{d}z \tag{6}$$

$$= 2\rho \int_{-h}^{h} \int_{0}^{a} \left[\frac{a^3}{\sqrt{3}} - \frac{4a^2 x}{\sqrt{3}} + 3\sqrt{3}ax^2 - 2\sqrt{3}x^3 \right] \mathrm{d}x\mathrm{d}z \tag{7}$$

$$= 2\rho \int_{-h}^{h} \left[\frac{a^3}{\sqrt{3}}x - \frac{2a^2x^2}{\sqrt{3}} + \sqrt{3}ax^3 - \frac{\sqrt{3}x^4}{2} \Bigg|_{0}^{a} \right] \mathrm{d}z \tag{8}$$

$$= 2\rho \int_{-h}^{h} \left[\frac{a^4}{\sqrt{3}} - \frac{2a^4}{\sqrt{3}} + \sqrt{3}a^4 \right] \mathrm{d}z \tag{9}$$

$$= 2\rho \int_{-h}^{h} \frac{\sqrt{3}a^4}{6} \mathrm{d}z = \frac{2\sqrt{3}}{3}\rho h a^4 \rightarrow \rho = \frac{m}{2\sqrt{3}a^2 h} \tag{10}$$

$$= \frac{2\sqrt{3}}{3} \frac{m}{2\sqrt{3}a^2 h} a^4 h = \frac{ma^2}{3} \tag{11}$$

(b) SOLUTION - Both products of inertia $\mathcal{I}_{XZ}$ and $\mathcal{I}_{YZ}$ are equal to zero, and it's possible to see that by setting up the integrals,

$$\mathcal{I}_{XZ} = \int_{-a}^{a} \int_{-h}^{h} xz\mathrm{d}z\mathrm{d}x = 0 \tag{12}$$

because this is an odd function with symmetric limits. Same result for a symmetry argument: considering just the x-y plane, both products of inertia are zero as z = 0 in this case. To calculate the integral would be the same as summing all the contribution of each cross section of the prism, varying z from -h to h. given that all contributions are zero, $\mathcal{I}_{XZ} = \mathcal{I}_{YZ} = 0$. Another symmetry argument: thw way this problem is set up, the z axis is a principal axis of rotation, which means $\mathcal{I}_{XZ} = \mathcal{I}_{YZ} = 0$ because $\mathcal{I}_{ZZ}$ is a principal moment of inertia, one of the values for the intertia tensor.

2. Problem 10.23 from Taylor

SOLUTION - Given that the sheet is on the xy plane, the elements $\mathcal{I}_{XZ}$ and $\mathcal{I}_{YZ}$ will be zero because $z = 0$ in this reference frame. Applying the same logic, we have:

$$\mathcal{I}_{ZZ} = \int (x^2 + y^2)\mathrm{d}m \tag{13}$$

$$\mathcal{I}_{XX} = \int (z^2 + y^2)\mathrm{d}m = \int y^2\mathrm{d}m \tag{14}$$

$$\mathcal{I}_{YY} = \int (x^2 + z^2)\mathrm{d}m = \int x^2\mathrm{d}m \tag{15}$$

$$\tag{16}$$

Thus

$$\mathcal{I}_{ZZ} = \mathcal{I}_{XX} + \mathcal{I}_{YY} \quad \square \tag{17}$$

3. Problem 10.30 and 10.41 from Taylor

(a) **SOLUTION** - Using the result from problem 10.23, we know the inertia tensor looks like this:

$$\mathcal{I} = \begin{pmatrix} \mathcal{I}_{XX} & \mathcal{I}_{YX} & 0 \\ \mathcal{I}_{XY} & \mathcal{I}_{YY} & 0 \\ 0 & 0 & \mathcal{I}_{ZZ} \end{pmatrix} \tag{18}$$

$$\tag{19}$$

and it's possible to see this choice of axis makes the tensor block diagonal, meaning $\mathcal{I}_{ZZ}$ is the eigenvalue. Another way to prove this is by calculating $\mathbf{L}$,

$$\mathbf{L} = \mathcal{I}\omega = \mathcal{I}_{ZZ}\omega \tag{20}$$

since the angular velocity is along the z axis, making $\mathbf{L}$ and ω parallel and the axis going through the origin a principal axis of rotation.

(b) **SOLUTION** - For problem 10.41, we need to prove $\frac{\mathrm{d}}{\mathrm{d}t}(\omega_1^2 + \omega_2^2) = 0$. Making use of the following equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} = (\omega_1^2 + \omega_2^2) = 2\omega_1\dot{\omega}_1 + 2\omega_2\dot{\omega}_2 = 0 \tag{21}$$

and the Euler's equations (no torque):

$$\begin{cases} \lambda_1\dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3 \\ \lambda_2\dot{\omega}_2 = (\lambda_3 - \lambda_1)\omega_3\omega_1 \end{cases} \tag{22}$$

where λ_1, λ_2 and λ_3 represent the principal moments of inertia, so from problem 10.23, $\lambda_3 = \lambda_1 + \lambda_2$.

Plugging that into Euler's equations:

$$\begin{cases} \lambda_1\dot{\omega_1} = (\lambda_2 - (\lambda_1 + \lambda_2))\omega_2\omega_3(\lambda_2 - \lambda_1 - \lambda_2)\omega_2\omega_3 = \lambda_1\omega_2\omega_3 \\ \lambda_2\dot{\omega_2} = (\lambda_1 + \lambda_2 - \lambda_1)\omega_3\omega_1 = \lambda_2\omega_3\omega_1 \end{cases} \tag{23}$$

Multiplying the frist equation by $2\omega_1$ and the second one by $2\omega_2$:

$$\begin{cases} \lambda_1(2\omega_1\dot{\omega_1}) = -\lambda_1\omega_2\omega_3\omega_1 \rightarrow 2\omega_1\dot{\omega_1} = \omega_1\omega_2\omega_3 \\ \lambda_2(2\omega_2\dot{\omega_2}) = -\lambda_2\omega_3\omega_1\omega_2 \rightarrow 2\omega_2\dot{\omega_2} = \omega_1\omega_2\omega_3 \end{cases} \tag{24}$$

so

$$2\omega_1\dot{\omega_1} + 2\omega_2\dot{\omega_2} = 0 \quad \square \tag{25}$$

4. Problem 10.45 from Taylor

(a) **SOLUTION** - Show that the free precession should have period 305 days. Starting with values given for Earth: $\lambda_1 = \lambda_2; \lambda_3 1.00327\lambda_1$, we calculate the precession first:

$$\Omega_b = \frac{\lambda_1 - \lambda_3}{\lambda_1}\omega_3 = \frac{\lambda_1(1 - 1.00327)}{\lambda_1}\frac{2\pi}{1\text{day}} = \frac{-0.00327(2\pi)}{1\text{day}}. \tag{26}$$

Then we calculate the period:

$$T = \frac{2\pi}{\Omega_b} = 305.81\text{days}. \tag{27}$$

The negative precession frequency only means that it's in the opposite direction of ω. From equation 10.118:

$$\Omega_s = \frac{\omega\sqrt{\lambda_3^2(\lambda_1^2 - \lambda_3^2)\sin\alpha}}{\lambda_1} \tag{28}$$

with $\alpha = 0.2$ arcseconds, or $\approx 10^{-7}$ rad. In this case, α is so small that we can approximate $\sin\alpha \approx 0$:

$$\Omega_s = \frac{\omega\sqrt{\lambda_3^2}}{\lambda_1} = \frac{1.00327(2\pi)}{1\text{day}} \tag{29}$$

$$T = \frac{2\pi}{\Omega_s} \approx 1\text{day} \quad \square \tag{30}$$