There are a total of 100 points on this exam. Attempt all problems. Partial credit will be given, wherever possible, for attempted problems provided that all the work is shown clearly. Just explaining first in words how you plan to do the problem could get you some credit but to get full credit you need to justify your answers.

You may use your book, your 12 graded problem sets plus their solution as provided, and the notes you made in class but no worked-out problems. You can use a calculator if necessary, but not read stored text from it. Write your name and ID on the exam, since it will be collected when the exam is over together with your answers.

Name:

University ID:

1) Kepler's third law is arrived at by employing the expression

$$\tau^2 = 4\pi^2 \frac{a^3 \mu}{\gamma}.$$

In deriving Kepler's third law we made an approximation based on the fact that the sun's mass M_s is much greater than that of planet m and we then obtained

$$\tau^2 = 4\pi^2 \frac{a^3}{G M_S}.$$

   a) (10 points) Show that the law should actually read

$$\tau^2 = 4\pi^2 \frac{a^3}{G(M_S + m)}$$

   b) (5 points) The "constant" of proportionality is therefore a little different for different planets. Given that Jupiter is about 2×10^{27} kg, while M_S is about 2×10^{30} kg (and some planets have masses several orders of magnitude less than Jupiter), by what percent does the "constant" in Kepler's third law vary among the planets?

2) Consider the relation

$$\left(\frac{d\boldsymbol{Q}}{dt}\right)_{S_0} = \left(\frac{d\boldsymbol{Q}}{dt}\right)_S + \boldsymbol{\Omega} \times \boldsymbol{Q}$$

relating the derivative of any vector quantity $\boldsymbol{Q}$ as measured in an inertial frame S_0 to the corresponding derivative in the rotating frame S.

   a) (5 points) Explain this relation between the derivatives of a vector $\boldsymbol{Q}$ in the two frames S_0 and S for the special case that $\boldsymbol{Q}$ is fixed in S.

   b) (10 points) Do the same for a vector $\boldsymbol{Q}$ that is fixed in the frame S_0 and compare your answer to part a).

3) Denote by $\mathbf{I}^{\mathbf{CM}}$ the moment of inertia tensor of a rigid body (mass M) about its CM. Denote by $\mathbf{I}$ the moment of inertia tensor of a rigid body (mass M) about about a point P displaced from the CM by $\Delta = (\xi, \eta, \zeta)$ as shown in Fig. 1.

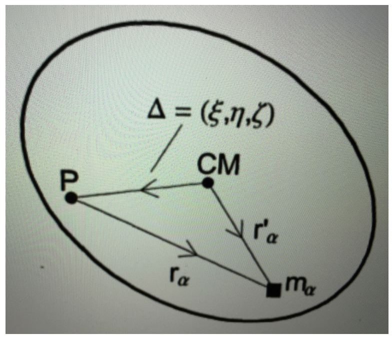

Figure 1: Picture for problem 3

a) (10 points) Prove that

$$I_{xx} = I_{xx}^{CM} + M(\eta^2 + \zeta^2).$$

b) (10 points) Also show that

$$I_{yz} = I_{yz}^{CM} - M\eta\zeta.$$

The other elements are then obtained by cyclic permutations.

4) The analysis of the free precession of a symmetric body was based on Euler's equations (see Sec.10.8). We can obtain the same results using Euler's angles as follows: Since $\boldsymbol{L}$ is constant you may as well choose the space axis $\hat{\boldsymbol{z}}$ such that $\boldsymbol{L} = L\hat{\boldsymbol{z}}$.

a) (5 points) Use

$$\hat{\boldsymbol{z}} = (\cos\theta)\boldsymbol{e}_3 - (\sin\theta)\boldsymbol{e}'_1$$

to write $\boldsymbol{L}$ in terms of $\boldsymbol{e}'_1, \boldsymbol{e}'_2$ and $\boldsymbol{e}_3$.

b) (5 points) By comparing with the result we obtained in class

$$\boldsymbol{L} = (-\lambda_1\dot{\phi}\sin\theta)\boldsymbol{e}'_1 + \lambda_1\dot{\theta}\boldsymbol{e}'_2 + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\boldsymbol{e}_3,$$

obtain three equations for $\dot{\phi}, \dot{\theta}$, and $\dot{\psi}$.

c) (10 points) Demonstrate that θ and $\dot{\phi}$ are constant, and that the rate of precession of the body axis about the space axis $\hat{\boldsymbol{z}}$ is $\Omega_s = L/\lambda_1$.

d) (10 points) Using

$$\boldsymbol{\omega} = (-\dot{\phi}\sin\theta)\boldsymbol{e}_1' + \dot{\theta}\boldsymbol{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\boldsymbol{e}_3$$

show that the angle between $\boldsymbol{\omega}$ and $\boldsymbol{e}_3$ is constant and that the three vectors $\boldsymbol{L}, \boldsymbol{\omega}$, and $\boldsymbol{e}_3$ are always coplanar.

5) Consider again the the double pendulum shown in Fig. 2.

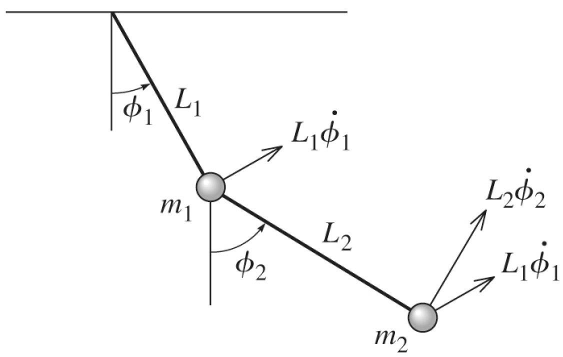

Figure 2: Picture for problem 5

a) (10 points) Write down the exact Lagrangian valid for all angles and find the corresponding equations of motion.

b) (10 points) Demonstrate that these equations reduce to the equations we discussed in class when both angles are small.