Development

- General state

$$|n_{\boldsymbol{k}_1\alpha_1} n_{\boldsymbol{k}_2\alpha_2} n_{\boldsymbol{k}_3\alpha_3} \ldots\rangle = \prod_{\boldsymbol{k}_i\alpha_i} \frac{\left(a^\dagger_{\boldsymbol{k}_i\alpha_i}\right)^{n_{\boldsymbol{k}_i\alpha_i}}}{\sqrt{n_{\boldsymbol{k}_i\alpha_i}!}} |0\rangle$$

- So that

$$a^\dagger_{\boldsymbol{k}_i\alpha_i} |n_{\boldsymbol{k}_1\alpha_1} \ldots n_{\boldsymbol{k}_i\alpha_i} \ldots\rangle = \sqrt{n_{\boldsymbol{k}_i\alpha_i} + 1}\, |n_{\boldsymbol{k}_1\alpha_1} \ldots (n_{\boldsymbol{k}_i\alpha_i} + 1) \ldots\rangle$$

- Photons: quantum excitations of the radiation field since classical vector potential has been replaced by quantum operator acting on photon states!

$$A_{\boldsymbol{k}\alpha} \Rightarrow -ic\sqrt{4\pi}\left[Q_{\boldsymbol{k}\alpha} + \frac{i}{\omega_k} P_{\boldsymbol{k}\alpha}\right] = \frac{c\sqrt{4\pi}}{\omega_k}\left[-i\omega_k Q_{\boldsymbol{k}\alpha} + P_{\boldsymbol{k}\alpha}\right] \frac{1}{\sqrt{2\hbar\omega_k}} \times \sqrt{2\hbar\omega_k}$$

$$= c\sqrt{\frac{8\pi\hbar}{\omega_k}} a_{\boldsymbol{k}\alpha}$$

- also $A^*_{\boldsymbol{k}\alpha} \Rightarrow c\sqrt{\frac{8\pi\hbar}{\omega_k}} a^\dagger_{\boldsymbol{k}\alpha}$

Vector potential operator

$$\boldsymbol{A}(\boldsymbol{x},t) = \sum_{\boldsymbol{k}\alpha} \left(\frac{2\pi\hbar c^2}{\omega_k V} \right)^{1/2} \left\{ a_{\boldsymbol{k}\alpha} \boldsymbol{e}_{\boldsymbol{k}\alpha} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_k t)} + a^{\dagger}_{\boldsymbol{k}\alpha} \boldsymbol{e}_{\boldsymbol{k}\alpha} e^{-i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_k t)} \right\}$$

Acts on photon states: adds or removes one!

Acts on charged particle at **x** and t (first quantization)

First rewrite Hamiltonian of free field for further interpretation

No work...

Hamiltonian free field

Number operator for each mode $\hat{N}_{\boldsymbol{k}\alpha} = a^\dagger_{\boldsymbol{k}\alpha} a_{\boldsymbol{k}\alpha}$

Hamiltonian operator $\hat{H}_{em} = \sum_{\boldsymbol{k}\alpha} \hbar\omega_k \left(\hat{N}_{\boldsymbol{k}\alpha} + \frac{1}{2}\right) \Rightarrow \sum_{\boldsymbol{k}\alpha} \hbar\omega_k \hat{N}_{\boldsymbol{k}\alpha}$

Momentum operator from Poynting vector (exercise)

$$\begin{aligned}\hat{\boldsymbol{P}}_{em} &= \frac{1}{8\pi c}\int_V d^3x\,(\boldsymbol{E}\times\boldsymbol{B} - \boldsymbol{B}\times\boldsymbol{E}) \\ &= \sum_{\boldsymbol{k}\alpha} \hbar\boldsymbol{k}\left(\hat{N}_{\boldsymbol{k}\alpha} + \frac{1}{2}\right) = \sum_{\boldsymbol{k}\alpha} \hbar\boldsymbol{k}\hat{N}_{\boldsymbol{k}\alpha}\end{aligned}$$

Single photon state $\hat{H}_{em} a^\dagger_{\boldsymbol{k}\alpha}|0\rangle = \hbar\omega_k\ a^\dagger_{\boldsymbol{k}\alpha}|0\rangle$

$$\hat{\boldsymbol{P}}_{em} a^\dagger_{\boldsymbol{k}\alpha}|0\rangle = \hbar\boldsymbol{k}\ a^\dagger_{\boldsymbol{k}\alpha}|0\rangle$$

So massless!

$$m^2c^4 = E^2 - \boldsymbol{p}^2c^2 = \hbar^2\omega_k^2 - \hbar^2k^2c^2 = \hbar^2k^2c^2 - \hbar^2k^2c^2 = 0$$

More on photon states

- Characterized also by polarization vector $\boldsymbol{e}_{\boldsymbol{k}\alpha}$
- Transforms as vector --> interpret as 1 unit of intrinsic angular momentum or spin of the photon
- Consider circular polarization vectors

$$\boldsymbol{e}_{\boldsymbol{k}}^{(\pm)} = \mp \frac{1}{\sqrt{2}} (\boldsymbol{e}_{\boldsymbol{k},1} \pm i \boldsymbol{e}_{\boldsymbol{k},2})$$

- Rotate by angle $\delta\phi$ about propagation axis

$$\begin{aligned} \boldsymbol{e}'_{\boldsymbol{k},1} &= \cos\delta\phi \; \boldsymbol{e}_{\boldsymbol{k},1} + \sin\delta\phi \; \boldsymbol{e}_{\boldsymbol{k},2} && \Rightarrow \boldsymbol{e}_{\boldsymbol{k},1} + \delta\phi \; \boldsymbol{e}_{\boldsymbol{k},2} \\ \boldsymbol{e}'_{\boldsymbol{k},2} &= -\sin\delta\phi \; \boldsymbol{e}_{\boldsymbol{k},1} + \cos\delta\phi \; \boldsymbol{e}_{\boldsymbol{k},2} && \Rightarrow -\delta\phi \; \boldsymbol{e}_{\boldsymbol{k},1} + \boldsymbol{e}_{\boldsymbol{k},2} \end{aligned}$$

- New circular polarization vectors

$$\begin{aligned} \boldsymbol{e}_{\boldsymbol{k}}^{\pm\prime} &= \mp \frac{1}{\sqrt{2}} (\boldsymbol{e}_{\boldsymbol{k},1'} \pm i \boldsymbol{e}_{\boldsymbol{k},2'}) \\ &= \boldsymbol{e}_{\boldsymbol{k}}^{(\pm)} \mp \frac{1}{\sqrt{2}} \delta\phi \, (\boldsymbol{e}_{\boldsymbol{k},2} \pm (-) i \boldsymbol{e}_{\boldsymbol{k},1}) \\ &= \boldsymbol{e}_{\boldsymbol{k}}^{(\pm)} \mp i\delta\phi \; \boldsymbol{e}_{\boldsymbol{k}}^{(\pm)} \\ &= (1 \mp i\delta\phi) \, \boldsymbol{e}_{\boldsymbol{k}}^{(\pm)} \end{aligned}$$

Angular momentum

- Compare $$\boldsymbol{e}_{\boldsymbol{k}}^{\pm\prime} = (1 \mp i\delta\phi)\, \boldsymbol{e}_{\boldsymbol{k}}^{(\pm)}$$
- With $$e^{-\frac{i}{\hbar} J_z \phi}\, |1m\rangle = e^{-im\phi}\, |1m\rangle$$ $$\Rightarrow (1 - im\delta\phi)\, |1m\rangle$$
- Interpret $$m = 1 \Rightarrow \boldsymbol{e}_{\boldsymbol{k}}^{(+)}$$ $$m = -1 \Rightarrow \boldsymbol{e}_{\boldsymbol{k}}^{(-)}$$
- Quantization axis along $\boldsymbol{k}$ so photons can have helicity 1 or -1 but not 0 --> no longitudinal photons
- No contradiction (no rest frame where photon is at rest)
- Photons with good helicity

$$a_{\boldsymbol{k}\pm}^{\dagger} = \mp \frac{1}{\sqrt{2}} \left(a_{\boldsymbol{k},1}^{\dagger} \pm i a_{\boldsymbol{k},2}^{\dagger} \right)$$

Interaction of electrons with photons

- Complete Hamiltonian includes interaction of charges and their coupling to the electromagnetic field
- Use radiation gauge
- Vector potential in minimal substitution
- Hamiltonian for Z electrons in an atom plus radiation field

$$
\begin{aligned}
H &= \sum_{i=1}^{Z} \frac{\left(\boldsymbol{p}_i + \frac{e}{c}\boldsymbol{A}(\boldsymbol{x}_i, t)\right)^2}{2m} - \sum_{i=1}^{Z} \frac{Ze^2}{|\boldsymbol{x}_i|} + \sum_{i<j}^{Z} \frac{e^2}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} + \hat{H}_{em} \\
&= H_{electrons} + H_{int} + \hat{H}_{em}
\end{aligned}
$$

$$
\hat{H}_{em} = \sum_{\boldsymbol{k}\alpha} \hbar\omega_k a^{\dagger}_{\boldsymbol{k}\alpha} a_{\boldsymbol{k}\alpha}
$$

Electron and interaction Hamiltonian

Electrons

$$H_{electrons} = \sum_{i=1}^{Z} \frac{\boldsymbol{p}_i^2}{2m} - \sum_{i=1}^{Z} \frac{Ze^2}{|\boldsymbol{x}_i|} + \sum_{i<j}^{Z} \frac{e^2}{|\boldsymbol{x}_i - \boldsymbol{x}_j|}$$

Coupling

$$H_{int} = \sum_{i}^{Z} \left[\frac{e}{2mc} \left(\boldsymbol{p}_i \cdot \boldsymbol{A}(\boldsymbol{x}_i, t) + \boldsymbol{A}(\boldsymbol{x}_i, t) \cdot \boldsymbol{p}_i \right) + \frac{e^2}{2mc^2} \boldsymbol{A}(\boldsymbol{x}_i, t) \cdot \boldsymbol{A}(\boldsymbol{x}_i, t) \right]$$

with

$$\boldsymbol{A}(\boldsymbol{x}_i, t) = \sum_{\boldsymbol{k}\alpha} \left(\frac{2\pi \hbar c^2}{\omega_k V} \right)^{1/2} \left\{ a_{\boldsymbol{k}\alpha} \boldsymbol{e}_{\boldsymbol{k}\alpha} e^{i(\boldsymbol{k}\cdot\boldsymbol{x}_i - \omega_k t)} + a^\dagger_{\boldsymbol{k}\alpha} \boldsymbol{e}_{\boldsymbol{k}\alpha} e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}_i - \omega_k t)} \right\}$$

No spin yet. Add by hand as before from $E = -\boldsymbol{\mu} \cdot \boldsymbol{B}$

$$H_{int}^{spin} = \frac{e}{mc} \sum_{i=1}^{Z} \boldsymbol{s}_i \cdot \left[\boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{x}, t) \right]_{\boldsymbol{x}=\boldsymbol{x}_i}$$