Inclusion of the electromagnetic field in Quantum Mechanics similar to Classical Mechanics but with interesting consequences

- Maxwell's equations
- Scalar and vector potentials
- Lorentz force
- Transform to Lagrangian
- Then Hamiltonian
- Minimal coupling to charged particles

Maxwell's equations

Gaussian units

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{x}, t) &= 4\pi\rho(\boldsymbol{x}, t) \\
\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{x}, t) &= 0 \\
\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{x}, t) &= -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{x}, t) \\
\boldsymbol{\nabla} \times \boldsymbol{B}(\boldsymbol{x}, t) &= \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{E}(\boldsymbol{x}, t) + \frac{4\pi}{c}\boldsymbol{j}(\boldsymbol{x}, t)
\end{aligned}$$

Scalar and Vector potential

Quantum applications require replacing

electric and **magnetic fields!**

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \quad \text{implies} \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

From Faraday $\boldsymbol{\nabla} \times \left(\boldsymbol{E} + \frac{1}{c} \frac{\partial}{\partial t} \boldsymbol{A} \right) = 0$

so $\boldsymbol{E} + \frac{1}{c} \frac{\partial}{\partial t} \boldsymbol{A} = -\boldsymbol{\nabla} \Phi$

or $\boldsymbol{E} = -\boldsymbol{\nabla} \Phi - \frac{1}{c} \frac{\partial \boldsymbol{A}}{\partial t}$

in terms of vector and scalar potentials.

Homogeneous equations are automatically solved.

Gauge freedom

Remaining equations using $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A}$

$$\begin{aligned} \boldsymbol{\nabla}^2 \Phi + \frac{1}{c} \frac{\partial}{\partial t} (\boldsymbol{\nabla} \cdot \boldsymbol{A}) &= -4\pi\rho \\ \boldsymbol{\nabla}^2 \boldsymbol{A} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} \right) &= -\frac{4\pi}{c} \boldsymbol{j} \end{aligned}$$

To decouple one could choose (gauge freedom)

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} = 0$$

more later... first

Coupling to charged particles

Lorentz $$\boldsymbol{F} = q\left\{\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right\}$$

Rewrite $$\boldsymbol{F} = q\left\{-\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} + \frac{1}{c}\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\right\}$$

Note $$\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A}) - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}$$

and $$\frac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A} = \frac{d\boldsymbol{A}}{dt}$$

So that $$\boldsymbol{F} = -\boldsymbol{\nabla}U + \frac{d}{dt}\frac{\partial U}{\partial \boldsymbol{v}}$$ with $$U = q\Phi - \frac{q}{c}\boldsymbol{v}\cdot\boldsymbol{A}$$

Check

Yields Lorentz from $L = T - U = \frac{1}{2}m\boldsymbol{v}^2 - q\Phi + \frac{q}{c}\boldsymbol{v}\cdot\boldsymbol{A}$

Equations of motion $$\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{v}} - \frac{\partial L}{\partial \boldsymbol{x}} = 0$$

Generalized momentum $$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A}$$

Solve for **v** and substitute in Hamiltonian

--> Hamiltonian for a charged particle

$$H = \boldsymbol{p}\cdot\boldsymbol{v} - L = \frac{\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)^2}{2m} + q\Phi$$

Include external electromagnetic field in QM

- Static electric field: nothing new (position --> operator)
- Include static magnetic field with momentum and position operators

$$H = \frac{\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}(\boldsymbol{x})\right)^2}{2m}$$

- Note velocity operator $$\boldsymbol{v} = \frac{1}{m}\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)$$
- Note Hamiltonian not "free" particle one
- Use $$[p_i, A_j] = \frac{\hbar}{i}\frac{\partial A_j}{\partial x_i}$$
- to show that $$[v_i, v_j] = i\frac{q\hbar}{m^2 c}\epsilon_{ijk}B_k$$
- Gauge independent! So think in terms of $H = \frac{1}{2}m|\boldsymbol{v}|^2$

Include external electromagnetic field

- Include uniform magnetic field
- For example by $\boldsymbol{B}(\boldsymbol{x}) = B\hat{\boldsymbol{z}}$
- Only nonvanishing commutator $[v_x, v_y] = i\dfrac{q\hbar B}{m^2 c}$
- Write Hamiltonian as

$$H = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right)$$

- but now $v_z = \dfrac{p_z}{m}$ so this corresponds to free particle motion parallel to magnetic field (true classically too)
- Only consider

$$H = \frac{1}{2}m\left(v_x^2 + v_y^2\right)$$

- Operators don't commute but commutator is a complex number!
- So...

Harmonic oscillator again...

- Motion perpendicular to magnetic field --> harmonic oscillator
- Introduce

$$a = \sqrt{\frac{m}{2\hbar\omega_c}}(v_x + iv_y)$$

$$a^\dagger = \sqrt{\frac{m}{2\hbar\omega_c}}(v_x - iv_y)$$

- with cyclotron frequency $\omega_c = \frac{qB}{mc}$
- Straightforward to check $\left[a, a^\dagger\right] = 1$
- So Hamiltonian becomes

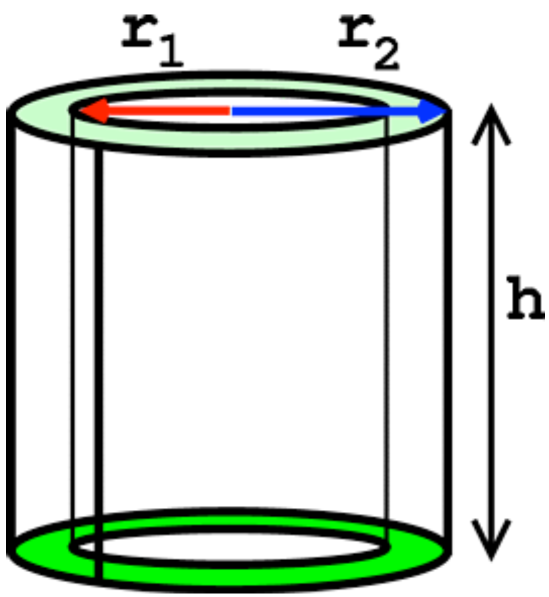
$$H = \hbar\omega_c\left(a^\dagger a + \tfrac{1}{2}\right)$$

- and consequently spectrum is (called Landau levels)

$$E_n = \hbar\omega_c\left(n + \tfrac{1}{2}\right) \qquad n = 0, 1, \ldots.$$

Aharanov-Bohm effect

- Consider hollow cylindrical shell



- Magnetic field inside inner cylinder either on or off
- Charged particle confined between inner and outer radius as well as top and bottom

Discussion

- Without field:
  - Wave function vanishes at the radii of the cylinders as well as top and bottom --> discrete energies
- With field (think of solenoid)
  - No magnetic field where the particle moves; inside in z-direction and constant
  - Spectrum changes because the vector potential is needed in the Hamiltonian
  - Use Stokes theorem $$\int_S (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \hat{\boldsymbol{n}}\, da = \oint_C \boldsymbol{A} \cdot d\boldsymbol{\ell}$$
  - Only z-component of magnetic field so left-hand side becomes $$\int_S (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \hat{\boldsymbol{n}}\, da = \int_S B\theta(r_1 - \rho) da = B\pi r_1^2$$
  - for any circular loop outside inner cylinder (and centered)
  - Vector potential in the direction of $\hat{\phi}$ and line integral --> $2\pi r$
  - Resulting in $\boldsymbol{A} = \frac{B r_1^2}{2r}\hat{\phi}$ modifying the Hamiltonian and the spectrum!!

Example

- No field
- Example of radial wave function
- Problem solved in cylindrical coordinates

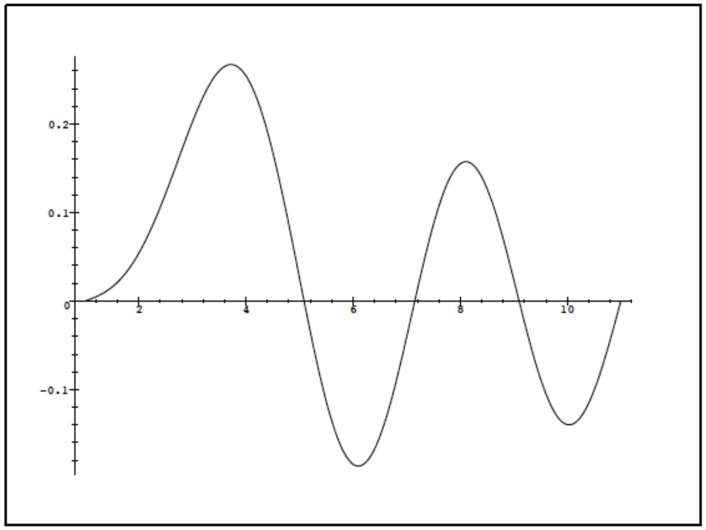


Figure 2: Radial eigenfunction for $n = 4$ and $f = 0$

- Also with field -->

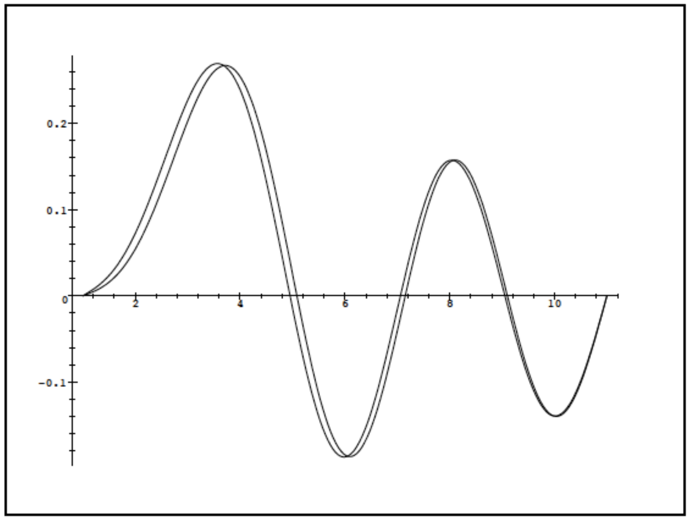


Figure 3: Radial eigenfunctions for $f = 0$ and $f = 0.4$

Quantize electromagnetic field

- Classical free field equations
- Quantize
- Photons
- Coupling to charged particles
- One-body operator acting on charged particles and photons

Maxwell's equations

Gaussian units

$$
\begin{aligned}
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\boldsymbol{\nabla} \times \boldsymbol{B}(\boldsymbol{x}, t) &= \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{E}(\boldsymbol{x}, t) + \frac{4\pi}{c}\boldsymbol{j}(\boldsymbol{x}, t)
\end{aligned}
$$

Scalar and Vector potential

Quantum applications require replacing **electric** and **magnetic fields!**

$$\begin{aligned} \boldsymbol{E} &= -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \\ \boldsymbol{B} &= \boldsymbol{\nabla} \times \boldsymbol{A} \end{aligned}$$

in terms of vector and scalar potentials.
Homogeneous equations are automatically solved.

Gauge freedom

Remaining equations

$$
\begin{aligned}
\boldsymbol{\nabla}^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) &= -4\pi\rho \\
\boldsymbol{\nabla}^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) &= -\frac{4\pi}{c}\boldsymbol{j}
\end{aligned}
$$

To decouple employ gauge freedom.

Observe: adding gradient of scalar function to vector potential yields same magnetic field

To keep electric field the same: change scalar potential accordingly!

Gauge transformation

- Explicitly

$$\boldsymbol{A} \Rightarrow \boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\Lambda$$

$$\Phi \Rightarrow \Phi' = \Phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}$$

- With

$$\boldsymbol{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$$

--> same E&M fields

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

- Can always find potentials that satisfy $\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0$
- If not: choose Λ such that

$$0 = \boldsymbol{\nabla} \cdot \boldsymbol{A}' + \frac{1}{c}\frac{\partial \Phi'}{\partial t} = \boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla}^2\Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial^2 t}$$

Employ this gauge freedom

$$\begin{aligned} \boldsymbol{\nabla}^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) &= -4\pi\rho \\ \boldsymbol{\nabla}^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) &= -\frac{4\pi}{c}\boldsymbol{j} \end{aligned}$$

Can choose $\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0$ (Lorentz gauge)

Leads to wave equations

$$\begin{aligned} \boldsymbol{\nabla}^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} &= -4\pi\rho \\ \boldsymbol{\nabla}^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} &= -4\pi\boldsymbol{j} \end{aligned}$$

Radiation gauge

$$\begin{aligned}\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{A}) &= -4\pi\rho \\ \nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \nabla\left(\nabla\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) &= -\frac{4\pi}{c}\boldsymbol{j}\end{aligned}$$

Alternative: radiation gauge (Coulomb, or transverse gauge)--> useful for quantizing free field

$$\nabla\cdot\boldsymbol{A} = 0$$

yields

$$\begin{aligned}\nabla^2\Phi &= -4\pi\rho \\ \nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} &= \frac{1}{c}\nabla\frac{\partial\Phi}{\partial t} - \frac{4\pi}{c}\boldsymbol{j}\end{aligned}$$

Instantaneous Coulomb

Yields instantaneous Coulomb potential $\Phi(\boldsymbol{x},t) = \int_V d^3x' \, \frac{\rho(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}$

Vector potential --> inhomogeneous wave equation

rhs can be calculated from instantaneous Coulomb potential

Now no sources $\Rightarrow$ free field

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

and $\boldsymbol{\nabla}^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$ $\Rightarrow$ solve in large box with volume $V = L^3$

Free field solutions

Use periodic BC so expand in plane waves to avoid standing ones

Allowed values $k_x = n_x \frac{2\pi}{L}$ $n_x = 0, \pm 1, \pm 2, \ldots$ also for y and z

Normalization $$\frac{1}{V}\int_V d\boldsymbol{x}\; e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}} = \delta_{\boldsymbol{k}\boldsymbol{k}'}$$

So solution can be written as $$\boldsymbol{A}(\boldsymbol{x},t) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}} \boldsymbol{A}_{\boldsymbol{k}}(t)\; e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Gauge choice $\Rightarrow$ $\boldsymbol{k}\cdot\boldsymbol{A}_{\boldsymbol{k}} = 0$

So for future reference: $$\boldsymbol{A}_{\boldsymbol{k}} = \sum_{\alpha=1,2} \boldsymbol{e}_{\boldsymbol{k}\alpha} A_{\boldsymbol{k}\alpha}$$ (polarizations)

From wave equation $$\frac{\partial^2 \boldsymbol{A}_{\boldsymbol{k}}(t)}{\partial t^2} + c^2 k^2 \boldsymbol{A}_{\boldsymbol{k}}(t) = 0$$ for each mode

Harmonic solutions

Fourier coefficients oscillate harmonically $\Rightarrow \omega_k = ck$

So time dependence: $\boldsymbol{A_k}(t) = e^{-i\omega_k t}\ \boldsymbol{A_k}$

Given initial distribution of $\boldsymbol{A_k}(t=0)$ --> problem solved!

E&M fields real so make vector potential explicitly real

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{x},t) &= \frac{1}{2\sqrt{V}}\left(\sum_{\boldsymbol{k}} \boldsymbol{A_k}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \sum_{\boldsymbol{k}} \boldsymbol{A}^*_{\boldsymbol{k}}(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right) \\ &= \frac{1}{2\sqrt{V}}\sum_{\boldsymbol{k}}\left[\boldsymbol{A_k}(t) + \boldsymbol{A}^*_{-\boldsymbol{k}}(t)\right]e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\end{aligned}$$

Fields

Use

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{1}{2\sqrt{V}}\sum_{\boldsymbol{k}}\left[\boldsymbol{A}_{\boldsymbol{k}}(t)+\boldsymbol{A}^*_{-\boldsymbol{k}}(t)\right]e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Then electric field

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x},t) &= -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\\ &= -\frac{1}{2c\sqrt{V}}\sum_{\boldsymbol{k}}\left(-i\omega_k\ \boldsymbol{A}_{\boldsymbol{k}}(t)+i\omega_k\ \boldsymbol{A}^*_{-\boldsymbol{k}}(t)\right)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\\ &= \frac{i}{2c\sqrt{V}}\sum_{\boldsymbol{k}}\omega_k\ \left[\boldsymbol{A}_{\boldsymbol{k}}(t)-\boldsymbol{A}^*_{-\boldsymbol{k}}(t)\right]e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\end{aligned}$$

and magnetic field

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{x},t) &= \boldsymbol{\nabla}\times\boldsymbol{A}\\ &= \frac{i}{2\sqrt{V}}\sum_{\boldsymbol{k}}\boldsymbol{k}\times\left[\boldsymbol{A}_{\boldsymbol{k}}(t)+\boldsymbol{A}^*_{-\boldsymbol{k}}(t)\right]e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\end{aligned}$$

Energy in field

General
$$H_{em} = \frac{1}{8\pi}\int_V d\boldsymbol{x}\ \left(\boldsymbol{E}\cdot\boldsymbol{E} + \boldsymbol{B}\cdot\boldsymbol{B}\right)$$

Use
$$\boldsymbol{E}(\boldsymbol{x},t) = \frac{i}{2c\sqrt{V}}\sum_{\boldsymbol{k}} \omega_k\ \left[\boldsymbol{A}_{\boldsymbol{k}}(t) - \boldsymbol{A}^*_{-\boldsymbol{k}}(t)\right] e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Note fields are real so

$$\begin{aligned}
\int_V d\boldsymbol{x}\ \boldsymbol{E}\cdot\boldsymbol{E} &= \int_V d\boldsymbol{x}\ \boldsymbol{E}\cdot\boldsymbol{E}^* \\
&= \int_V d\boldsymbol{x}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'}\frac{1}{4c^2V}\omega_k\left(\boldsymbol{A}_{\boldsymbol{k}}(t) - \boldsymbol{A}^*_{-\boldsymbol{k}}(t)\right)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\
&\quad \times\cdot\,\omega_{k'}\left(\boldsymbol{A}^*_{\boldsymbol{k}'}(t) - \boldsymbol{A}_{-\boldsymbol{k}'}(t)\right)e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}
\end{aligned}$$

Orthogonality
$$\begin{aligned}
&= \frac{1}{4c^2}\sum_{\boldsymbol{k}}\omega_k^2\left|\boldsymbol{A}_{\boldsymbol{k}}(t) - \boldsymbol{A}^*_{-\boldsymbol{k}}(t)\right|^2 \\
&= \frac{1}{4}\sum_{\boldsymbol{k}} k^2\left|\boldsymbol{A}_{\boldsymbol{k}}(t) - \boldsymbol{A}^*_{-\boldsymbol{k}}(t)\right|^2
\end{aligned}$$

Energy in field continued

Similarly (exercise)

$$\int_V dx\, \boldsymbol{B}\cdot\boldsymbol{B} = \int_V dx\, \boldsymbol{B}\cdot\boldsymbol{B}^* = \frac{1}{4}\sum_{\boldsymbol{k}} k^2 \left|\boldsymbol{A}_{\boldsymbol{k}}(t) + \boldsymbol{A}^*_{-\boldsymbol{k}}(t)\right|^2$$

So with

$$\int_V dx\, \boldsymbol{E}\cdot\boldsymbol{E} = \frac{1}{4}\sum_{\boldsymbol{k}} k^2 \left|\boldsymbol{A}_{\boldsymbol{k}}(t) - \boldsymbol{A}^*_{-\boldsymbol{k}}(t)\right|^2$$

Energy becomes

$$H_{em} = \frac{1}{8\pi}\int_V dx\, \left(\boldsymbol{E}\cdot\boldsymbol{E} + \boldsymbol{B}\cdot\boldsymbol{B}\right) = \frac{1}{8\pi}\frac{1}{4}\sum_{\boldsymbol{k}} 2k^2 \left(\left|\boldsymbol{A}_{\boldsymbol{k}}(t)\right|^2 + \left|\boldsymbol{A}_{-\boldsymbol{k}}(t)\right|^2\right) = \frac{1}{8\pi}\sum_{\boldsymbol{k}} k^2 \left|\boldsymbol{A}_{\boldsymbol{k}}(t)\right|^2 = \frac{1}{8\pi}\sum_{\boldsymbol{k}} k^2 \left|\boldsymbol{A}_{\boldsymbol{k}}\right|^2$$

Note: no time dependence!

Expand Fourier coefficients along polarization vectors

Use $$\boldsymbol{A}_{\boldsymbol{k}} = \sum_{\alpha=1,2} \boldsymbol{e}_{\boldsymbol{k}\alpha} A_{\boldsymbol{k}\alpha}$$

--> $$H_{em} = \frac{1}{8\pi} \sum_{\boldsymbol{k}\alpha} k^2 \; |A_{\boldsymbol{k}\alpha}|^2$$

Preparation for QUANTIZATION

In order to quantize, introduce real canonical variables

$$\boldsymbol{Q}_{\boldsymbol{k}}(t) = \frac{i}{2c\sqrt{4\pi}}\left[\boldsymbol{A}_{\boldsymbol{k}}(t) - \boldsymbol{A}^*_{\boldsymbol{k}}(t)\right]$$

$$\boldsymbol{P}_{\boldsymbol{k}}(t) = \frac{k}{2\sqrt{4\pi}}\left[\boldsymbol{A}_{\boldsymbol{k}}(t) + \boldsymbol{A}^*_{\boldsymbol{k}}(t)\right]$$

Invert --> $\boldsymbol{A}_{\boldsymbol{k}}(t) = -ic\sqrt{4\pi}\left[\boldsymbol{Q}_{\boldsymbol{k}}(t) + \frac{i}{\omega_k}\boldsymbol{P}_{\boldsymbol{k}}(t)\right]$

So $$|\boldsymbol{A}_{\boldsymbol{k}}(t)|^2 = c^2 4\pi\left[\boldsymbol{Q}^2_{\boldsymbol{k}}(t) + \frac{\boldsymbol{P}^2_{\boldsymbol{k}}(t)}{\omega_k^2}\right] = c^2 4\pi\left[\boldsymbol{Q}^2_{\boldsymbol{k}} + \frac{\boldsymbol{P}^2_{\boldsymbol{k}}}{\omega_k^2}\right]$$

And thus .....(what else)

Oscillators of course

$$
\begin{aligned}
H_{em} &= \frac{1}{8\pi}\sum_{\boldsymbol{k}} k^2 \left|\boldsymbol{A}_{\boldsymbol{k}}(t)\right|^2 \\
&= \frac{1}{8\pi}\sum_{\boldsymbol{k}} k^2 c^2 4\pi \left(\boldsymbol{Q}_{\boldsymbol{k}}^2 + \frac{\boldsymbol{P}_{\boldsymbol{k}}^2}{\omega_k^2}\right) \\
&= \frac{1}{2}\sum_{\boldsymbol{k}} \left(\boldsymbol{P}_{\boldsymbol{k}}^2 + \omega_k^2 \boldsymbol{Q}_{\boldsymbol{k}}^2\right)
\end{aligned}
$$

Expand in polarizations $\boldsymbol{P}_{\boldsymbol{k}} = \sum_{\alpha=1,2} \boldsymbol{e}_{\boldsymbol{k}\alpha} P_{\boldsymbol{k}\alpha} \quad \boldsymbol{Q}_{\boldsymbol{k}} = \sum_{\alpha=1,2} \boldsymbol{e}_{\boldsymbol{k}\alpha} Q_{\boldsymbol{k}\alpha}$

then $$H_{em} = \frac{1}{2}\sum_{\boldsymbol{k}\alpha} \left(P_{\boldsymbol{k}\alpha}^2 + \omega_k^2 Q_{\boldsymbol{k}\alpha}^2\right)$$

True canonical variables

$Q_{\boldsymbol{k}}, P_{\boldsymbol{k}}$ are canonical variables

Check
$$\boldsymbol{A}_{\boldsymbol{k}}(t) = e^{-i\omega_k t}\boldsymbol{A}_{\boldsymbol{k}}$$

So
$$\dot{\boldsymbol{A}}_{\boldsymbol{k}}(t) = -i\omega_k \boldsymbol{A}_{\boldsymbol{k}}(t)$$

and from
$$\boldsymbol{Q}_{\boldsymbol{k}}(t) = \frac{i}{2c\sqrt{4\pi}}\left[\boldsymbol{A}_{\boldsymbol{k}}(t) - \boldsymbol{A}^*_{\boldsymbol{k}}(t)\right]$$

it follows
$$\dot{\boldsymbol{Q}}_{\boldsymbol{k}}(t) = \frac{i}{2c\sqrt{4\pi}}\left[-i\omega_k \boldsymbol{A}_{\boldsymbol{k}}(t) - (i\omega_k)\boldsymbol{A}^*_{\boldsymbol{k}}(t)\right] = \boldsymbol{P}_{\boldsymbol{k}}(t)$$

But also:
$$\dot{\boldsymbol{Q}}_{\boldsymbol{k}} = \boldsymbol{P}_{\boldsymbol{k}} = \frac{\partial H_{em}}{\partial \boldsymbol{P}_{\boldsymbol{k}}}$$

Similarly for generalized momentum
$$\dot{\boldsymbol{P}}_{\boldsymbol{k}} = -\frac{\partial H_{em}}{\partial \boldsymbol{Q}_{\boldsymbol{k}}}$$

And now….

- Back to Hamiltonian
- Looks like a sum of oscillators --> treat as such!

- From canonical classical variables in classical mechanics

- Quantize by introducing commutation relations between operators!!! (Dirac)

$$
\begin{aligned}
[P_{\boldsymbol{k}\alpha}, P_{\boldsymbol{k}'\alpha'}] &= 0 \\
[Q_{\boldsymbol{k}\alpha}, Q_{\boldsymbol{k}'\alpha'}] &= 0 \\
[Q_{\boldsymbol{k}\alpha}, P_{\boldsymbol{k}'\alpha'}] &= i\hbar\delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\alpha,\alpha'}
\end{aligned}
$$

Photons

Introduce the usual operators

$$a_{\boldsymbol{k}\alpha} = \frac{1}{\sqrt{2\hbar\omega_k}}\left(P_{\boldsymbol{k}\alpha} - i\omega_k Q_{\boldsymbol{k}\alpha}\right)$$

$$a^\dagger_{\boldsymbol{k}\alpha} = \frac{1}{\sqrt{2\hbar\omega_k}}\left(P_{\boldsymbol{k}\alpha} + i\omega_k Q_{\boldsymbol{k}\alpha}\right)$$

with commutators

$$[a_{\boldsymbol{k}\alpha}, a_{\boldsymbol{k}'\alpha'}] = 0$$

$$\left[a^\dagger_{\boldsymbol{k}\alpha}, a^\dagger_{\boldsymbol{k}'\alpha'}\right] = 0$$

$$\left[a_{\boldsymbol{k}\alpha}, a^\dagger_{\boldsymbol{k}'\alpha'}\right] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\alpha,\alpha'}$$

Each mode HO

Number operator for each mode $\hat{N}_{\boldsymbol{k}\alpha} = a^\dagger_{\boldsymbol{k}\alpha} a_{\boldsymbol{k}\alpha}$

Then

$$\begin{aligned}\left[a_{\boldsymbol{k}\alpha}, \hat{N}_{\boldsymbol{k}'\alpha'}\right] &= a_{\boldsymbol{k}\alpha} a^\dagger_{\boldsymbol{k}'\alpha'} a_{\boldsymbol{k}'\alpha'} - a^\dagger_{\boldsymbol{k}'\alpha'} a_{\boldsymbol{k}'\alpha'} a_{\boldsymbol{k}\alpha} \\ &= a_{\boldsymbol{k}\alpha} a^\dagger_{\boldsymbol{k}'\alpha'} a_{\boldsymbol{k}'\alpha'} - a^\dagger_{\boldsymbol{k}'\alpha'} a_{\boldsymbol{k}\alpha} a_{\boldsymbol{k}'\alpha'} \\ &= \left[a_{\boldsymbol{k}\alpha} a^\dagger_{\boldsymbol{k}'\alpha'} - a^\dagger_{\boldsymbol{k}'\alpha'} a_{\boldsymbol{k}\alpha}\right] a_{\boldsymbol{k}'\alpha'} \\ &= \delta_{\boldsymbol{k}\boldsymbol{k}'} \delta_{\alpha\alpha'} a_{\boldsymbol{k}\alpha}\end{aligned}$$

and

$$\left[a^\dagger_{\boldsymbol{k}\alpha}, \hat{N}_{\boldsymbol{k}'\alpha'}\right] = -\delta_{\boldsymbol{k}\boldsymbol{k}'} \delta_{\alpha\alpha'} a^\dagger_{\boldsymbol{k}\alpha}$$

So enough to work with one mode $\hat{N} = a^\dagger a$

Eigenkets of this Hermitian operator $\hat{N}\left|n\right\rangle = n\left|n\right\rangle$

Consider $\hat{N} a^\dagger \left|n\right\rangle = \left[a^\dagger \hat{N} + a^\dagger\right]\left|n\right\rangle = (n+1) a^\dagger \left|n\right\rangle$

also eigenket with eigenvalue $n+1$

More

- Similarly $$\hat{N}a|n\rangle = \left[a\hat{N} - a\right]|n\rangle = (n-1)a|n\rangle$$
- So
$$\begin{aligned} a^\dagger|n\rangle &= c_+|n+1\rangle \\ a|n\rangle &= c_-|n-1\rangle \end{aligned}$$
- Normalization from
$$n = \langle n|\hat{N}|n\rangle = \langle n|a^\dagger a|n\rangle \geq 0$$
- Phase choice $$a|n\rangle = \sqrt{n}|n-1\rangle$$
- Also $$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$
- Integers otherwise negative norm appears
$$\begin{aligned} a|n\rangle &= \sqrt{n}|n-1\rangle \\ a|n-1\rangle &= \sqrt{n-1}|n-2\rangle \\ &\cdots \\ a|2\rangle &= \sqrt{2}|1\rangle \\ a|1\rangle &= \sqrt{1}|0\rangle \\ a|0\rangle &= 0 \end{aligned}$$

Photon states

- Operator that adds a photon with momentum $\hbar \boldsymbol{k}$ and polarization α

$$a^\dagger_{\boldsymbol{k}\alpha}$$

- Single photon state

$$a^\dagger_{\boldsymbol{k}\alpha} |0\rangle = |0, 0, ..., 0, 1_{\boldsymbol{k}\alpha}, 0........\rangle = |1_{\boldsymbol{k}\alpha}\rangle$$

- No quantum: vacuum state $|0\rangle$
- Normalized two-photon state (same mode)

$$\frac{1}{\sqrt{2}} a^\dagger_{\boldsymbol{k}\alpha} a^\dagger_{\boldsymbol{k}\alpha} |0\rangle = |0, 0, ..., 0, 2_{\boldsymbol{k}\alpha}, 0........\rangle = |2_{\boldsymbol{k}\alpha}\rangle$$

- Different modes

$$a^\dagger_{\boldsymbol{k}\alpha} a^\dagger_{\boldsymbol{k}'\alpha'} |0\rangle = |0, 0, ..., 0, 1_{\boldsymbol{k}\alpha}, 0..., 0, 1_{\boldsymbol{k}'\alpha'}, 0.....\rangle = |1_{\boldsymbol{k}\alpha} 1_{\boldsymbol{k}'\alpha'}\rangle = a^\dagger_{\boldsymbol{k}'\alpha'} a^\dagger_{\boldsymbol{k}\alpha} |0\rangle$$

Development

- General state

$$|n_{\boldsymbol{k}_1\alpha_1} n_{\boldsymbol{k}_2\alpha_2} n_{\boldsymbol{k}_3\alpha_3} \ldots\rangle = \prod_{\boldsymbol{k}_i\alpha_i} \frac{\left(a^\dagger_{\boldsymbol{k}_i\alpha_i}\right)^{n_{\boldsymbol{k}_i\alpha_i}}}{\sqrt{n_{\boldsymbol{k}_i\alpha_i}!}} |0\rangle$$

- So that

$$a^\dagger_{\boldsymbol{k}_i\alpha_i} |n_{\boldsymbol{k}_1\alpha_1} \ldots n_{\boldsymbol{k}_i\alpha_i} \ldots\rangle = \sqrt{n_{\boldsymbol{k}_i\alpha_i} + 1}\, |n_{\boldsymbol{k}_1\alpha_1} \ldots (n_{\boldsymbol{k}_i\alpha_i} + 1) \ldots\rangle$$

- Photons: quantum excitations of the radiation field since classical vector potential has been replaced by quantum operator acting on photon states!

$$A_{\boldsymbol{k}\alpha} \Rightarrow -ic\sqrt{4\pi}\left[Q_{\boldsymbol{k}\alpha} + \frac{i}{\omega_k} P_{\boldsymbol{k}\alpha}\right] = \frac{c\sqrt{4\pi}}{\omega_k}\left[-i\omega_k Q_{\boldsymbol{k}\alpha} + P_{\boldsymbol{k}\alpha}\right] \frac{1}{\sqrt{2\hbar\omega_k}} \times \sqrt{2\hbar\omega_k}$$

$$= c\sqrt{\frac{8\pi\hbar}{\omega_k}}\, a_{\boldsymbol{k}\alpha}$$

- also $A^*_{\boldsymbol{k}\alpha} \Rightarrow c\sqrt{\frac{8\pi\hbar}{\omega_k}}\, a^\dagger_{\boldsymbol{k}\alpha}$

Vector potential operator

$$\boldsymbol{A}(\boldsymbol{x},t)=\sum_{\boldsymbol{k}\alpha}\left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2}\left\{a_{\boldsymbol{k}\alpha}\boldsymbol{e}_{\boldsymbol{k}\alpha}e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_k t)}+a^\dagger_{\boldsymbol{k}\alpha}\boldsymbol{e}_{\boldsymbol{k}\alpha}e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_k t)}\right\}$$

Acts on photon states: adds or removes one!

Acts on charged particle at **x** and t (first quantization)

First rewrite Hamiltonian of free field for further interpretation

No work...

Hamiltonian free field

Number operator for each mode $\hat{N}_{\boldsymbol{k}\alpha} = a^\dagger_{\boldsymbol{k}\alpha} a_{\boldsymbol{k}\alpha}$

Hamiltonian operator $\hat{H}_{em} = \sum_{\boldsymbol{k}\alpha} \hbar\omega_k \left(\hat{N}_{\boldsymbol{k}\alpha} + \frac{1}{2}\right) \Rightarrow \sum_{\boldsymbol{k}\alpha} \hbar\omega_k \hat{N}_{\boldsymbol{k}\alpha}$

Momentum operator from Poynting vector (exercise)

$$\begin{aligned} \hat{\boldsymbol{P}}_{em} &= \frac{1}{8\pi c} \int_V d^3x \, (\boldsymbol{E} \times \boldsymbol{B} - \boldsymbol{B} \times \boldsymbol{E}) \\ &= \sum_{\boldsymbol{k}\alpha} \hbar\boldsymbol{k} \left(\hat{N}_{\boldsymbol{k}\alpha} + \tfrac{1}{2}\right) = \sum_{\boldsymbol{k}\alpha} \hbar\boldsymbol{k} \hat{N}_{\boldsymbol{k}\alpha} \end{aligned}$$

Single photon state $\hat{H}_{em} a^\dagger_{\boldsymbol{k}\alpha} |0\rangle = \hbar\omega_k \; a^\dagger_{\boldsymbol{k}\alpha} |0\rangle$

$$\hat{\boldsymbol{P}}_{em} a^\dagger_{\boldsymbol{k}\alpha} |0\rangle = \hbar\boldsymbol{k} \; a^\dagger_{\boldsymbol{k}\alpha} |0\rangle$$

So massless!

$$m^2c^4 = E^2 - \boldsymbol{p}^2c^2 = \hbar^2\omega_k^2 - \hbar^2k^2c^2 = \hbar^2k^2c^2 - \hbar^2k^2c^2 = 0$$

More on photon states

- Characterized also by polarization vector $\boldsymbol{e}_{\boldsymbol{k}\alpha}$
- Transforms as vector --> interpret as 1 unit of intrinsic angular momentum or spin of the photon
- Consider circular polarization vectors

$$\boldsymbol{e}_{\boldsymbol{k}}^{(\pm)} = \mp \frac{1}{\sqrt{2}} \left(\boldsymbol{e}_{\boldsymbol{k},1} \pm i \boldsymbol{e}_{\boldsymbol{k},2} \right)$$

- Rotate by angle $\delta\phi$ about propagation axis

$$\begin{aligned} \boldsymbol{e}'_{\boldsymbol{k},1} &= \cos\delta\phi \; \boldsymbol{e}_{\boldsymbol{k},1} + \sin\delta\phi \; \boldsymbol{e}_{\boldsymbol{k},2} && \Rightarrow \boldsymbol{e}_{\boldsymbol{k},1} + \delta\phi \; \boldsymbol{e}_{\boldsymbol{k},2} \\ \boldsymbol{e}'_{\boldsymbol{k},2} &= -\sin\delta\phi \; \boldsymbol{e}_{\boldsymbol{k},1} + \cos\delta\phi \; \boldsymbol{e}_{\boldsymbol{k},2} && \Rightarrow -\delta\phi \; \boldsymbol{e}_{\boldsymbol{k},1} + \boldsymbol{e}_{\boldsymbol{k},2} \end{aligned}$$

- New circular polarization vectors $\boldsymbol{e}_{\boldsymbol{k}}^{\pm\prime}$

$$\begin{aligned} \boldsymbol{e}_{\boldsymbol{k}}^{\pm\prime} &= \mp \frac{1}{\sqrt{2}} \left(\boldsymbol{e}_{\boldsymbol{k},1'} \pm i \boldsymbol{e}_{\boldsymbol{k},2'} \right) \\ &= \boldsymbol{e}_{\boldsymbol{k}}^{(\pm)} \mp \frac{1}{\sqrt{2}} \delta\phi \left(\boldsymbol{e}_{\boldsymbol{k},2} \pm (-) i \boldsymbol{e}_{\boldsymbol{k},1} \right) \\ &= \boldsymbol{e}_{\boldsymbol{k}}^{(\pm)} \mp i\delta\phi \; \boldsymbol{e}_{\boldsymbol{k}}^{(\pm)} \\ &= (1 \mp i\delta\phi) \, \boldsymbol{e}_{\boldsymbol{k}}^{(\pm)} \end{aligned}$$

Angular momentum

- Compare $$\boldsymbol{e}_{\boldsymbol{k}}^{\pm\prime} = (1 \mp i\delta\phi)\, \boldsymbol{e}_{\boldsymbol{k}}^{(\pm)}$$
- With $$e^{-\frac{i}{\hbar} J_z \phi} \left|1m\right\rangle = e^{-im\phi} \left|1m\right\rangle$$
$$\Rightarrow (1 - im\delta\phi) \left|1m\right\rangle$$
- Interpret $$m = 1 \Rightarrow \boldsymbol{e}_{\boldsymbol{k}}^{(+)}$$
$$m = -1 \Rightarrow \boldsymbol{e}_{\boldsymbol{k}}^{(-)}$$
- Quantization axis along $\boldsymbol{k}$ so photons can have helicity 1 or -1 but not 0 --> no longitudinal photons
- No contradiction (no rest frame where photon is at rest)
- Photons with good helicity

$$a^{\dagger}_{\boldsymbol{k}\pm} = \mp \frac{1}{\sqrt{2}} \left(a^{\dagger}_{\boldsymbol{k},1} \pm i a^{\dagger}_{\boldsymbol{k},2} \right)$$